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CHAPTER I.

EXERCISES IN GEOMETRICAL INVENTION.

SECTION I.

THEOREMS IN SPECIAL OR ELEMENTARY GEOMETRY.

623. This chapter will afford a review of Parts I. and II., while it will greatly extend the student's knowledge of geometrical facts. Great pains should be taken to secure good habits as to neatness of execution in the construction of figures, orderly and proper arrangement of thought, and in style of expression. The practice of constructing every figure upon geometrical principles—guessing at nothing—cannot be too strongly commended. As to the *form* of a geometrical argument, observe the following order:

1st. The enunciation of the theorem or problem in general terms.

2d. The elucidation of the general statement, by reference to the particular figure which it is proposed to use.

3d. A description of the figure, with reference to any auxiliary construction which is used in the demonstration or solution.

4th. The demonstration proper.

---

624. If two adjacent sides of a quadrilateral are equal each to each, and the other two adjacent sides equal each to each, the diagonals intersect at right angles.

Sug's.—1st. Draw a quadrilateral having such sides as the data require, and draw its diagonals. 2d. State the proposition with reference to the figure.

3d. [In this case the regular third step is not required, as no auxiliary lines are necessary.] 4th. Prove that the diagonals are at right angles to each other. The demonstration is based upon a corollary in Section I., Part II., Chapter I.

625. Cor.—One of the diagonals is *bisected.* [State which one, and show why.]

---

626. If a parallelogram has one oblique angle, all its angles are oblique; and if it has one right angle, all its angles are right angles.

Sug's.—Let the student be careful to follow the order as heretofore given. No auxiliary construction is needed. The demonstration is based upon the doctrine of parallels.

---

627. The sum of three straight lines drawn from any point within a triangle to the vertices is less than the sum, and greater than the half sum of the three sides of the triangle.

Sug's.—The first statement is proved from (***276***) and the second from (***274.***)

---

628. A line drawn from any angle of a triangle to the middle of the opposite side, is less than the half sum of the adjacent sides, and greater than the difference between this half sum and half the third side.

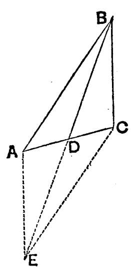

Fig. 356.

Sug's.—1st. Draw a triangle, as ABC, bisect one side, as AC, and draw BD. 2d. Make the statement with reference to the figure. 3d. Produce BD until DE = BD, and draw AE and EC. 4th. The first step in the proof is to show the triangle ADE equal to CBD, and ADB equal to DCE; whence AE = BC, and EC = AB.

---

629. If lines be drawn from the extremities of either of the oblique sides of a trapezoid to the middle of the opposite side, the triangle thus formed is half the trapezoid.

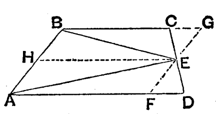

Fig. 357.

Sug's.—The third step, or construction, consists in drawing HE parallel to AD and hence to BC (?), and FG through E parallel to AB.

---

630. Any line drawn through the centre of the diagonal of a parallelogram bisects the figure.

631. Prove that the sum of the angles of a triangle is two right angles, by producing two of the sides about an angle and through this angle drawing a line parallel to the third side.

Prove the same by producing one side of the triangle and drawing a line through the exterior angle parallel to the non-adjacent side.

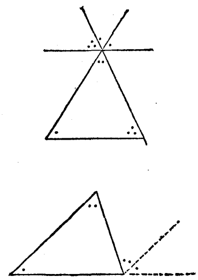

FIG. 358.

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632. If any point, not the centre, be taken in a diameter of a circle, of all the chords which can pass through that point, that one is the least which is at right angles to the diameter.

---

633. If from any point there extend two lines tangent to a circumference, the angle contained by the tangents is double the angle contained by the line joining the points of tangency and the radius extending to one of them.

---

634. The angle included by two lines drawn from any angle of a triangle, the one bisecting the angle and the other perpendicular to the opposite side, is half the difference of the other two angles of the triangle.

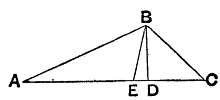

FIG. 359.

SUG'S. $ABD = 90° - A$, whence $ABD - EBD = 90° - A - EBD$. Also, $DBC = 90° - C$, whence $EBC = 90° - C + EBD = 90° - A - EBD$, etc.

---

635. If three lines be drawn from the acute angles of a right angled triangle—two bisecting these angles, and a third a perpendicular to one of the bisecting lines—the triangle included by these lines will be isosceles.

SUG'S.—It is to be proved that $OD = CD$. $COD = OAC + ACO = 45°$, etc.

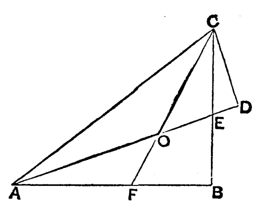

FIG. 360.

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636. If one circumference be described on the radius of another as a diameter, any straight line extending from their point of contact to the outer circumference is bisected by the inner.

SUG.—The demonstration is based upon (***159, 211***).

637. Prove that the sum of the angles of a regular five point star (Fig. 101) is two right angles. Show, also, that the figure formed by the intercepted portions of the lines is a regular pentagon.

---

638. If the sides of a regular hexagon are produced till they meet, show that the exterior figures will be equilateral triangles.

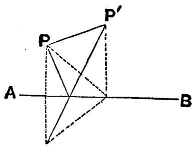

FIG. 361.

639. If from two given points on the same side of a given line, two lines be drawn meeting in the line, their sum is least when they make equal angles with the line.

---

640. If from two given points without a circumference, two lines be drawn meeting in the circumference, their sum is least when they make equal angles with a tangent at the common point.

---

641. The side of an equilateral triangle inscribed in a circle is equal to the diagonal of a rhombus, whose other diagonal and each of whose sides are equal to the radius.

---

642. If two circumferences intersect each other, and from either point of intersection a diameter be drawn in each, the other extremities of these diameters and the other point of intersection are in the same straight line.

---

643. If any straight line joining two parallels be bisected, any other line through the point of bisection and included by the parallels, is bisected at the same point.

---

644. If the sides of any quadrilateral are bisected, the quadrilateral formed by joining the adjacent points of bisection is a parallelogram.

SUG'S.—1st. Draw a quadrilateral, bisect its sides, and join the adjacent points of bisection. 2d. State the proposition, with reference to the figure. 3d. Draw the diagonals. 4th. Give the proof. It is based on the similarity of triangles.

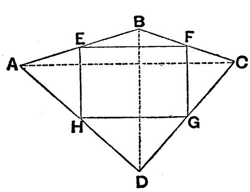

FIG. 362.

645. COR. 1.—The parallelogram is one-half the trapezium. Prove it. What figure is formed by joining the centres of EF, FG, and FG, HG, etc. ?

646. Cor. 2.—Lines joining the middle points of the opposite sides of any trapezium bisect each other (?).

---

647. If two straight lines join the alternate ends of two parallels, the line joining their centres is half the difference of the parallels.

Sug's.—We are to prove that $EF = \frac{1}{2}(CD - AB)$. $CH = EF = \frac{1}{2}(CD - AB)$.

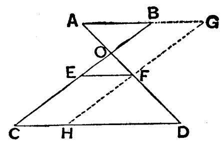

Fig. 363.

---

648. In any right-angled triangle the line drawn from the right angle to the middle of the hypotenuse is equal to one-half the hypotenuse.

---

649. The perpendiculars which bisect the three sides of a triangle meet in a common point.

Sug's.—First show that the intersection of two of the perpendiculars is equally distant from the three vertices of the triangle. Then that a line drawn from this point to the middle of the third side is perpendicular to it.

---

650. The three perpendiculars drawn from the angles of a triangle upon the opposite sides intersect in a common point.

Sug's.—Draw through the vertices of the triangle lines parallel to the opposite sides. The proposition may then be brought under the preceding.

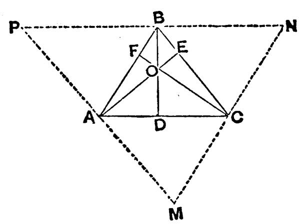

Fig. 364.

651. Cor.—The following triangles are similar—viz., BOE, BDC, AOD, and AEC, each to each; also BOF, BDA, DOC, and CFA. Prove it.

---

652. If from a point without a circle two secants be drawn, making equal angles with a third secant passing through the centre of the circle, the intercepted chords of the first two are equal.

Sug.—Prove by revolving one part of the figure.

653. The sum of the alternate angles of *any* hexagon inscribed in a circle is four right angles.

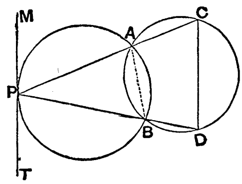

FIG. 365.

654. If two circles intersect in A and B, and from P, any point in one circumference, the chords PA and PB be drawn to cut the other in C and D, CD is parallel to a tangent at P.

---

655. If two lines intersect, two lines which bisect the opposite angles are perpendicular to each other.

---

656. The angle included by two lines drawn from a point within a triangle to the vertices of two of the angles, is greater than the third angle.

SUG'S.—The demonstration may be founded on (***219***) or (***231***).

---

657. In a triangle whose angles are 90°, 60°, and 30°, show that the longest side is twice the shortest.

---

658. Lines which bisect the adjacent angles of a parallelogram are mutually perpendicular.

---

659. If from any point in the base of an isosceles triangle lines are drawn parallel to the sides, a parallelogram is formed whose perimeter is constant and equal to the sum of the two equal sides of the triangle.

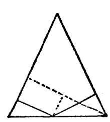

FIG. 366.

660. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides of the triangle, their sum is *constant* and equal to the perpendicular from one of the equal angles of the triangle upon the opposite side.

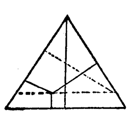

FIG. 367.

661. If from any point within an equilateral triangle, three perpendiculars be let fall upon the sides, their sum is constant and equal to the altitude of the triangle.

662. If from a fixed point without a circle two tangents be drawn terminating in the circumference, the triangle formed by them and any tangent to the included arc has a constant perimeter equal to the sum of the first two tangents.

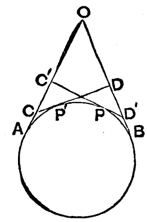

FIG. 368.

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663. The sum of two opposite sides of a quadrilateral circumscribed about a circle, is equal to the sum of the other two.

---

664. If two opposite angles of a quadrilateral are supplementary, it may be circumscribed by a circumference.

---

665. The square described on the sum of two lines is equivalent to the sum of the squares on the lines, *plus* twice the rectangle of the lines.

SUG'S.—Be careful to give the construction fully, and show that the parts are rectangles, etc.

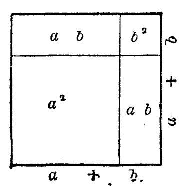

FIG. 369.

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666. The square described on the difference of two lines is equivalent to the sum of the squares on the lines, *minus* twice the rectangle of the lines.

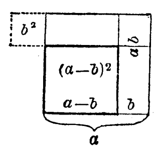

FIG. 370.

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667. The rectangle of the sum and difference of two lines is equivalent to the difference of the squares described on the lines.

SCH.—The three preceding propositions are but geometrical conceptions and demonstrations of the algebraic *formulæ*, $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, and $(a + b)(a - b) = a^2 - b^2$.

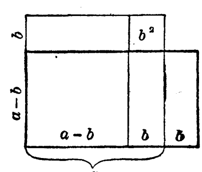


FIG. 371.

VARIOUS DEMONSTRATIONS OF THE PYTHAGOREAN PROPOSITION.

668. The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.

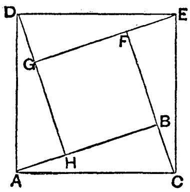


Fig. 372.

1st Method.—Let ABC be the given triangle, and ACED the square described on the hypotenuse. Complete the construction. Show that the four triangles are equal. The square HF is $(AB - BC)^2$. The student can complete the demonstration.

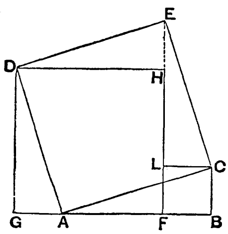


Fig. 373.

2d Method.—Let ACED be the square on the hypotenuse. Let fall the perpendiculars EF, DG, etc. Show that the three triangles are equal, and that FD and LB are the squares of the two sides AB and BC.

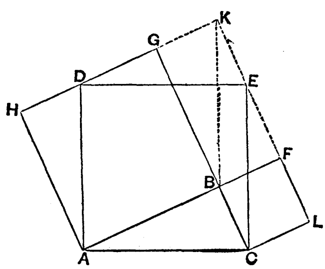


Fig. 374.

3d Method.—Let BL and BH be the squares on the sides. Produce FL and HG till they meet in K. Draw DA and EC perpendicular to AC, and draw DE and KB. Prove that ACED is a square, and also that the triangles ABC, CLE, BFK, KBG, DKE, and AHD are all equal to each other. The demonstration is then readily made.

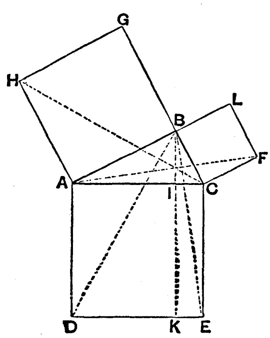
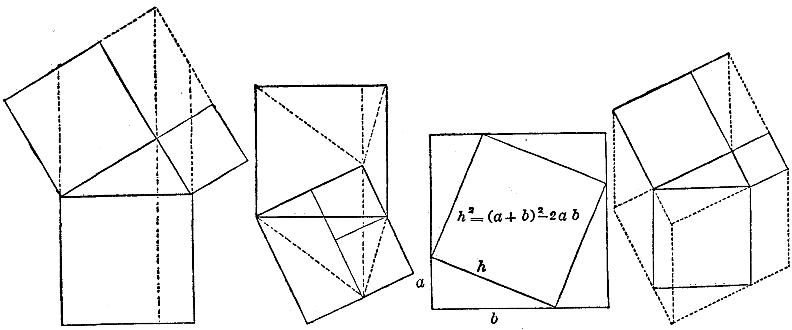


Fig. 375.

4th Method.—This is the demonstration usually given in our text-books. Drawing the squares on the three sides, let fall BI perpendicular to AC and produce it to K. Draw BD, BE, HC and AF. Show that the triangle HAC = BAD, and that the former is half the square AG, and the latter half the rectangle AK. Hence AG = AK. In like manner show that LC = CK.

We will now give a few other figures by means of which the demonstration can be effected, and leave the student to his own resources in effecting it.



5th METHOD. 6th METHOD. 7th METHOD. 8th METHOD.

FIG. 376.

9th METHOD.—The truth of the theorem appears also as a direct consequence of (*360*).

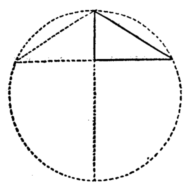


FIG. 377.

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669. In an oblique angled triangle the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides diminished by twice the rectangle of the base, and the distance from the acute angle to the foot of the perpendicular let fall upon the base from the angle opposite.

SUG'S.—It is to be shown that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2AC \times DC$. Observe that $\overline{AD}^2 = (AC - DC)^2 = \overline{AC}^2 + \overline{DC}^2 - 2AC \times DC$. Whence, by a simple application of the preceding theorem, the truth of this becomes apparent.

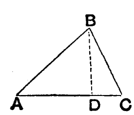

FIG. 378.

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670. In an obtuse angled triangle the square of the side opposite the obtuse angle is equivalent to the sum of the squares on the other two sides, increased by twice the rectangle contained by the base and the distance from the obtuse angle to the foot of the perpendicular let fall from the angle opposite upon the base produced.

SUG.—The demonstration is analogous to the preceding, C being made obtuse in this case; whence $AD = AC + DC$, etc.

671. The following is an outline of a general demonstration covering the three preceding propositions:

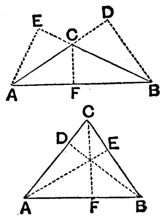

Fig. 379.

Letting AE, BD, and CF be the three perpendiculars from the angles upon the opposite sides, and observing that a circumference described on any side as a diameter passes through the feet of two of the perpendiculars, (***356***) and (***355***) readily give the following :

$$AB \times AF = AC \times AD = \overline{AC}^2 \pm AC \times CD,$$

$$\text{and } AB \times BF = BC \times BE = \overline{BC}^2 \pm BC \times CE;$$

adding, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 \pm 2AC \times CD$ (or $2BC \times CE$), the + sign being taken when C is obtuse, and the — sign when C is acute. If C is right CE and CD become 0, whence $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

672. Def.—The line drawn from any angle of a triangle to the middle of the opposite side is called a *medial line.*

673. The sum of the squares of any two sides of a triangle is equivalent to twice the square of the medial line drawn from their included angle, plus twice the square of half the third side.

Sug.—Proved by applying (***669, 670***).

674. The three medial lines of a triangle mutually trisect each other, and hence intersect in a common point.

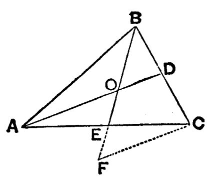

Fig. 380.

Sug's.—To prove that $OE = \frac{1}{3}BE$, draw FC parallel to AD until it meets BE produced. Then the triangles AEO and FEC are equal (?); whence EF = OE. Also, BO = OF (?).

Having shown that $OE = \frac{1}{3}BE$, by a similar construction we can show that $OD = \frac{1}{3}AD$.

Finally, we may show that the medial line from C to AB cuts off $\frac{1}{3}$ of BE, and hence cuts BE at the same point as does AD.

Another Dem.—Lines through O parallel to the sides trisect the sides, etc.

675. In any quadrilateral the sum of the squares of the sides is equivalent to the sum of the squares of the diagonals, plus four times the square of the line joining the centres of the diagonals.

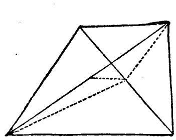

Fig. 381.

676. Cor.—The sum of the squares of the sides of a parallelogram is equivalent to the sum of the squares of the diagonals.

677. In any quadrilateral which may be inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides.

---

678. In any triangle the rectangle of two sides is equivalent to the rectangle of the perpendicular let fall from their included angle upon the third side, into the diameter of the circumscribed circle.

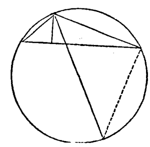
FIG. 382.

SUG.—This proposition is an immediate consequence of the similarity of two triangles in the figure.

679. COR.—The area of a triangle is equivalent to the product of its sides divided by twice the diameter of the circumscribed circle.

---

680. If there be an isosceles and an equilateral triangle on the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer one and from the ends of the base, then, according as the isosceles triangle is the inner or the outer one, its base angle will be $\frac{1}{4}$ of, or $2\frac{1}{2}$ times the vertical angle.

---

681. Of all triangles on the same base, and having their vertices in the same line parallel to the base, the isosceles has the greatest vertical angle.

SUG'S.—Circumscribe a circle about the isosceles triangle. By what is the vertical angle measured when the triangle is isosceles? By what, when it is not?

---

682. Two triangles are similar, when two sides of one are proportional to two sides of the other, and the angle opposite to that side which is equal to or greater than the other given side in one, is equal to the homologous angle in the other.

ALGEBRAIC DEMONSTRATIONS.

683. The difference of the squares on any side of a regular pentagon and any side of regular decagon, inscribed in the same circle, is equivalent to the square of the radius.

SUG'S.—We will give the outline of what may be termed an *Algebraic Demonstration* of this proposition. This method is often the most convenient and ex-

peditious. Letting p represent a side of the pentagon, d a side of the decagon, and r the radius, the student should be able to discover the following relations:

$$(1)\quad r : d :: d : r - d, \text{ or } r^2 - dr = d^2;$$

$$(2)\quad \sqrt{d^2 - \tfrac{1}{4}p^2} + \sqrt{r^2 - \tfrac{1}{4}p^2} = r.$$

From (2), $2r\sqrt{r^2 - \frac{1}{4}p^2} = 2r^2 - d^2 = r^2 + dr$, by substituting for d^2 its value from (1). Hence $4r^2 - p^2 = r^2 + 2dr + d^2$, or $3r^2 - 2dr = p^2 + d^2$. In this, substituting the value of dr as found in (1), we readily obtain $r^2 = p^2 - d^2$.

Q. E. D.

---

684. Demonstrate *algebraically* that the square on the sum of two lines, together with the square on the difference, is double the sum of the squares on the lines separately.

---

685. The sum of the squares of the three medial lines of a triangle is three-fourths of the sum of the squares of the sides.

---

686. The square of any side of a triangle is equivalent to twice the sum of the squares of the segments of the medial lines adjacent to its extremities, minus the square of the non-adjacent segment of the third medial line.

Deduced algebraically from the preceding.

---

687. The sum of the squares of the three greater segments of the medial lines of a triangle is equivalent to one-third the sum of the squares of the sides of a triangle.

Deduced algebraically from the preceding.

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688. The lines from the vertices of a triangle to the points of tangency of the inscribed circle intersect in a common point.

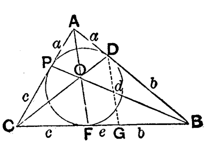

FIG. 383.

SUG'S. DC is parallel to AF, BD $=$ BF $= b$, CF $=$ CP $= c$, AD $=$ AP $= a$, DG $= d$, FG $= e$. OF $= \dfrac{dc}{c+e}$, $d = \dfrac{\text{AF} \times b}{a+b}$, $e = \dfrac{ab}{a+b}$. $\therefore$ OF $=$ AF $\times \dfrac{bc}{a(b+c)+bc}$. In like manner we may find where PB intersects AF, by drawing through P a parallel to AF. This distance is found to be OF $=$ AF $\times \dfrac{bc}{a(b+c)+bc}$, a result which might have been anticipated, since b and c are similarly involved.

689. The area of a triangle, as expressed in terms of its sides, is *Area = the square root of the continued product of half the sum of the sides into this half sum minus each side separately.*

SUG'S.—We will give the outlines of both the Geometric and Algebraic demonstrations:

1st. GEOMETRIC DEMONSTRATION. CD = CB, CE = CA, and through F, the centre of DA, HG is drawn parallel to AB. With F as a centre, and FH as a radius, a circumference passes through G (?). CN is perpendicular to AE and passes through H (?).

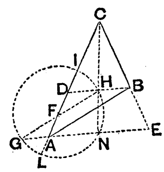

FIG. 384.

Now CF = $\frac{1}{2}$ (AC + CB), and FL = $\frac{1}{2}$AB (?).

Hence CL = $\frac{1}{2}$(AC + CB + AB) = $\frac{1}{2}$S, S being the sum of the sides;

Hence, also, DL = AI = $\frac{1}{2}$S − CB, CI = $\frac{1}{2}$S − AB, and AL = $\frac{1}{2}$S − AC.

Again, CN × AN = area ACE;

and HN × AN = area ABE;

Subtracting, AN (CN − HN) = AN × CH = area ACB (1).

Once more, CH × DH = area CDB;

and HN × DH = area ADB.

Adding, DH (CH + HN) = CA × CN = area ACB (2).

Multiplying (1) and (2), we have

CA × AN × CH × CN = (area ACB)².

But CN × CH = CL × CI, and CA × AN = AL × AI = AL × DL.

Therefore, *area* ACB = $\sqrt{\text{CL} \times \text{CI} \times \text{AL} \times \text{DL}}$ =

$\sqrt{\frac{1}{2}\text{S}\,(\frac{1}{2}\text{S} - \text{AB})\,(\frac{1}{2}\text{S} - \text{AC})\,(\frac{1}{2}\text{S} - \text{CB})}$.

2d. ALGEBRAIC DEMONSTRATION. From the right angled triangles BCD and ACD, we find $m = \frac{a^2 - b^2 + c^2}{2c}$.

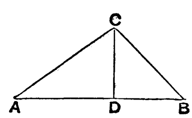

FIG. 385.

Whence $p = \sqrt{a^2 - \left(\frac{a^2 - b^2 + c^2}{2c}\right)^2}$, and

$$\text{area ABC} = \frac{c}{2}\sqrt{a^2 - \left(\frac{a^2 - b^2 + c^2}{2c}\right)^2}$$

$$= \frac{1}{4}\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}$$

$$= \frac{1}{4}\sqrt{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}$$

$$= \sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s - c)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - a)}.$$

690. From any point in the plane of a circle the greatest and least distances to the circumference are measured on the line passing through the centre.

Sug's.—There are three cases :—1st. When the point is without the circle. 2d. When the point is within. 3d. When the point is in the circumference.

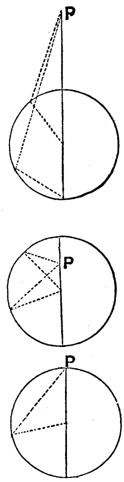

Fig. 386.

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691. From any point except the centre of a circle, two, and only two, equal lines can be drawn to the circumference.

Sug.—This is a direct consequence of (***181***, ***182***).

---

692. If two opposite sides of a parallelogram be bisected, straight lines from the points of bisection to the opposite vertices will trisect the diagonal.

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693. The feet of two perpendiculars let fall from two given points upon a given line are equally distant from the middle of the line joining the points.

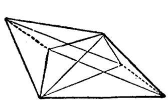

Fig. 387.

694. Two quadrilaterals are equivalent when their diagonals are respectively equal, and form equal angles.

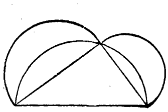

Fig. 388.

695. If, on the hypotenuse and sides of a right angled triangle, semicircumferences be described, that upon the hypotenuse passing through the vertex, the sum of the crescents thus formed will be equal to the area of the triangle.

696. The bisectors of any two exterior angles of a triangle meet in a point which is the centre of a circle, to which one side of the triangle and the other two produced are tangents.

These circles are called the *escribed circles*.

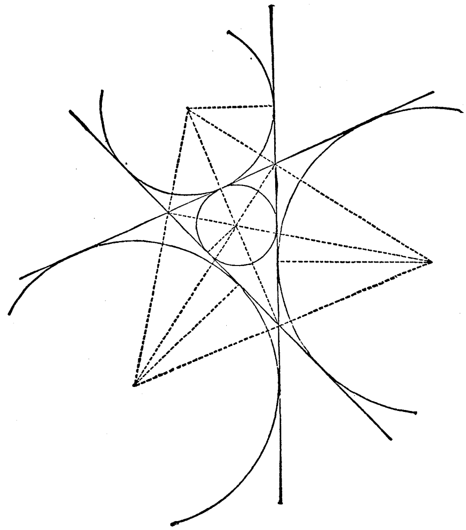

FIG. 389.

THE NINE POINTS CIRCLE.

697. In any triangle the centres of the THREE sides, the feet of the THREE perpendiculars from the vertices upon the opposite sides or sides produced, and the THREE middle points of the distances from the vertices to the common intersection of the perpendiculars, are NINE points in the circumference of one and the same circle; the centre of this circle is at the middle of the line joining the centre of the circumscribed circle and the common intersection of the perpendiculars; and the radius is half the radius of the circumscribed circle.

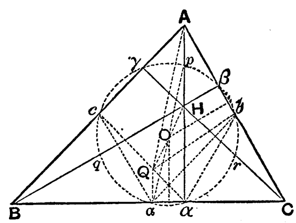

FIG. 390.

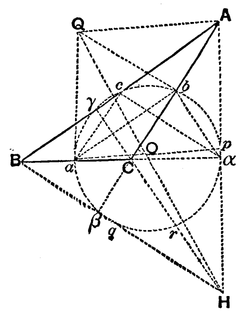

FIG. 391.

SUG'S.—The student will do well to confine his attention in the first instance to the first figure, and after he sees the demonstration in this case—*i. e.*, when the perpendiculars fall within, to trace it in the case of an obtuse angled triangle, in which the perpendiculars fall on the sides produced

1st. To show that the circle which passes through a, b, and c, also passes through α, we show the following relations among the angles: $cab = cAb =$

$c\mathsf{A}\alpha + \alpha\mathsf{A}b = cab$. Hence, the vertices a and α are in the same circumference. In like manner we show that β and γ are in the circumference passing through a, b, and c.

2d. Considering one of the partial triangles as BHC, α, β, and γ are the feet of the three perpendiculars from *its* vertices upon one of its sides and the prolongation of the other two. Therefore, by the first part r and q are the middle points of CH and BH. Considering either of the other partial triangles we find p the centre of AH.

3d. $a\alpha$ and $b\beta$ being chords of the nine points circle, O is its centre, and letting Q be the centre of the circumscribed circle, we may readily show that O is *in* QH, and also is at its middle point.

4th. Drawing aO, and producing it, we may show that it intersects AH in p, and hence $p\mathsf{H} = \mathsf{A}p = \mathsf{Q}a$, and AQ$ap$ is a parallelogram. Therefore $\mathsf{O}p = \frac{1}{2}pa = \frac{1}{2}\mathsf{QA}$.

698. Cor.—The middle points of the three lines joining the centres, two and two, of the escribed circles of a triangle, and the middle points of the three lines joining the centres of the escribed circles with the centre of the inscribed circle, are six points in the circumference of the circle circumscribed about the same triangle.

699. If one triangle be inscribed in another, the circumferences circumscribing the three exterior triangles thus formed intersect in a common point.

Sug.—The demonstration is founded on the property of the opposite angles of an inscribed quadrilateral. The construction lines extend from the vertices of the inscribed triangle to the intersection to be examined.

700. The difference between the hypotenuse and the sum of the other two sides of a right angled triangle is equal to the diameter of the inscribed circle.

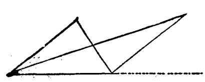
Fig. 392.

701. If from the extremities of any side of a triangle two lines be drawn, one bisecting an interior and the other an exterior angle, these lines will meet if sufficiently produced, and their included angle will be half the third angle of the triangle.

702. An inscribed equilateral triangle is one-fourth the circumscribed equilateral triangle about the same circle.

703. The three altitudes of a triangle are to each other inversely as the sides upon which they fall.

704. The bisectors of the angles included by the opposite sides of an inscribed quadrilateral intersect at right angles.

SUG.—By means of (***214***) show that FC + CE + HA + AG = 180°. Whence FOE = 90°.

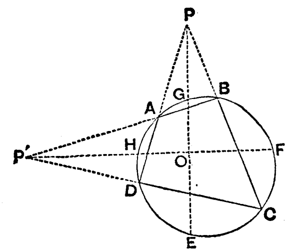

FIG. 393.

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705. Two triangles which have an angle in each equal, are to each other as the rectangle of the sides including the equal angle.

SUG'S. A and D being equal, we are to show that ABC : DEF :: AB × AC : DE × DF. Take AE′ = DE, AF′ = DF, and draw E′F′. Now from the facts that the triangles AE′F′ and DEF are equal, and that triangles of the same altitudes are to each other as their bases, the proposition is proved.

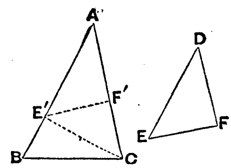

FIG. 394

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706. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equivalent, the figure is a trapezoid.

---

707. The difference between the angles which a medial line in a triangle makes with the side to which it is drawn, is equal to the difference of the angles of the triangle including this side.

---

708. If any number of equal right lines radiate from a common point, making equal consecutive angles, and any line be drawn through the common point, the sum of the perpendiculars upon this line from the extremities of the radiating lines on one side, is equal to the sum of those on the other side.

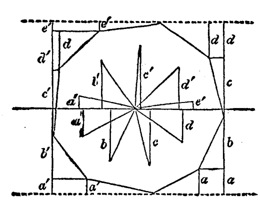

FIG. 395.

709. COR.—In any regular polygon, the sum of the perpendiculars let fall from the vertices on the one side of any line passing through the centre, is equal to the sum of those let fall from the vertices on the other side.

710. If the sum of two opposite sides of a quadrilateral is equal to the sum of the other two opposite sides, a circle may be inscribed in it.

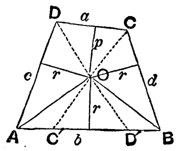

FIG. 396.

SUG'S.—Bisect any two adjacent angles, as A and B. Then are the perpendiculars r, r, r, equal (?); and it remains to be shown that the perpendicular $p = r$. Take AD′ = AD, and BC′ = BC, and draw OD′ and OC′. Since $a + b = c + d$, and $C'D' = b - (b - c) - (b - d)$, $C'D' = a$, and the triangles DOC and D′OC′ are equal. Hence $p = r$.

711. If two planes are parallel, any right line which pierces one, pierces the other also.

SUG.—Proof based on (***410***).

712. If two planes are parallel, any plane which intersects one, intersects the other also, and the lines of intersection are parallel.

713. COR.—Two planes which are parallel to a third, are parallel to each other.

714. A plane which is perpendicular to a line of another plane, or to a line parallel to that plane, is itself perpendicular to the latter plane.

715. If a straight line is perpendicular to a plane, any line parallel to the plane is perpendicular to the first line.

SUG.—Two lines in space which are not in the same plane, are said to make the same angle with each other as two lines respectively parallel to them and both lying in one plane.

716. In order that a straight line be perpendicular to a plane, it is sufficient that it be perpendicular to two lines not parallel to each other, and situated in the plane or parallel to it.

717. If two right lines in space are perpendicular to each other (not necessarily intersecting), their projections on a plane parallel to either line are perpendicular to each other.

SUG.—The *Projection* here referred to is that which is called the *Orthographic Projection.* The proposition is not generally true of the *Perspective Projection*, *i. e.*, the spaces which the lines (considered as material) would appear to occupy if they were placed between the eye and the plane. (See Ex. 8, page 174, PART II.)

718. The angle of inclination (***392,*** PART II.) of a line oblique to a plane, is less than the angle included between this line and any line of the plane, except its projection, which passes through the point in which the first line pierces the plane.

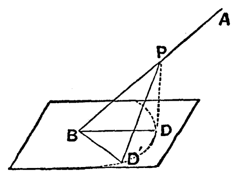

FIG. 397.

SUG. BD being the projection of AB, ABD < ABD′, BD′ being any line other than BD, passing through B.

---

719. Between any two lines not in the same plane, one line, and only one, can be drawn, which shall be perpendicular to both the given lines.

SUG'S.—Pass a plane through one of the lines parallel to the other; and through the other line pass a plane perpendicular to the first line.

---

720. In a warped quadrilateral, *i. e.*, one whose sides do not all lie in the same plane, the middle points of the sides are in one plane, and are the vertices of the angles of a parallelogram.

SUG.—Conceive the planes of two opposite angles of the quadrilateral, the intersection of which will be a diagonal of the given quadrilateral.

---

721. A line being given in a plane, one plane can be drawn including the given line and perpendicular to the first plane, and only one. Hence all right diedrals are equal.

SUG.—Demonstration similar to (***390***).

---

722. The plane angle formed by drawing two lines in the faces of a diedral, from a common point in the edge, is less than the measure of the diedral if the lines lie on the same side of the plane of the measure, and greater if they lie on opposite sides.

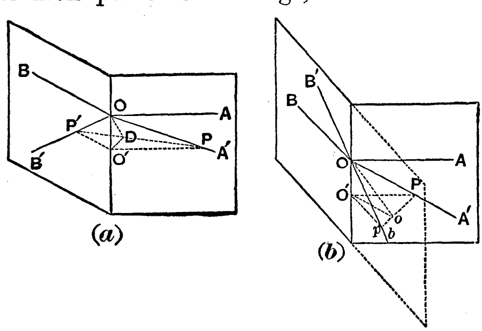

FIG. 398.

SUG'S.—In (*a*), AOB being the measure of the diedral, A′OB′ < AOB. Pass a plane through the diedral, perpendicular to its edge, and let its inter-

sections with A′O and B′O be P and P′, and with the edge, O′. Also, pass a plane through the edge and perpendicular to PP′. Revolve the triangle POP′ on PP′ till DO falls in DO′, etc.

The second case is demonstrated from (*b*).

---

723. If the projections of a line in the two faces of a diedral are straight, the line is a straight line.

Sug.—Proof based on (***386***).

---

724. If from the vertex of a triedral a line be drawn at pleasure within the triedral, the sum of the plane angles formed by this line and any two edges is less than the sum of the facial angles formed by the other edge and these two.

---

725. If through a point in space two lines be drawn parallel to a given plane, and through the same point two planes be passed respectively perpendicular to the two lines, the intersection of these two planes will be perpendicular to the given plane.

---

726. The three planes which bisect the three diedrals of a triedral intersect in a common line.

---

727. In any convex polyedral, the sum of the diedrals is greater than the sum of the angles of a polygon having the same number of sides that the polyedral has faces.

Sug.—Proof based upon (***722***).

---

728. Def.—***A Polyedron*** is a solid bounded by plane surfaces. A *Regular Convex Polyedron* is a polyedron whose faces are all equal regular polygons, and each of whose solid angles is convex outward, and is enclosed by the same number of faces.

---

729. There are five and only five regular convex polyedrons—viz.: *The Tetraedron,* whose faces are four equal equilateral triangles; *The Hexaedron,* or *Cube,* whose faces are six equal squares; *The Octaedron,* whose faces are eight equal equilateral triangles; *The Dodecaedron,* whose faces are twelve equal regular pentagons; and *The Icosaedron,* whose faces are twenty equal equilateral triangles.

Dem.—We demonstrate this proposition by showing—1st, that such solids can be constructed ; and 2d, that no others are possible.

The Regular Tetraedron.—Taking three equal equilateral triangles, as ASB, ASC, and BSC, it is possible to enclose a solid angle, as S, with them, since the sum of the three facial angles is (what ?) (Part II., ***436***). Then, since AC = AB = CB (?), considering ACB the fourth face, we have a regular polyedron whose four faces are equilateral triangles.

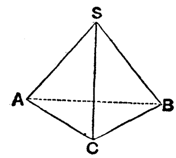

Fig. 399.

The Regular Hexaedron or *Cube.*—This is a familiar solid, but for purposes of uniformity and completeness we may conceive it constructed as follows : Taking three equal squares, as ASCB, CSED, and ASEF, we can enclose a solid angle, as S, with them (?). Now, conceive the planes of CB and CD, AB and AF, EF and ED produced. The plane of CB and CD being parallel to ASEF (?) will intersect the plane of EF and ED in HD parallel to FE (?). In like manner FH can be shown parallel to ED, BH to CD, and HD to BC. Hence the solid has for its faces six equal squares.

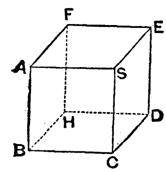

Fig. 400.

The Regular Octaedron.—At the intersection P, of the diagonals of a square, ABCD, erect a perpendicular SP to the plane of the square, and making SP = AP (half of one of the diagonals) draw SA, SD, SC, and SB. Making a similar construction on the other side of the plane ABCD, we have a solid having for faces eight equal equilateral triangles.

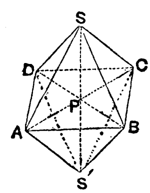

Fig. 401.

The Regular Dodecaedron.—Taking twelve equal regular pentagons, it is evident that we may group them in two sets of six each, as in the figure. Thus, around O we may place five, forming 5 triedrals at the vertices of O. These triedrals are possible, since the sum of the facial angles enclosing each is $3\frac{3}{5}$ right angles (?) —*i. e.*, between 0 and 4 right angles (Part II., ***436***). In like manner the other 6 may be grouped by placing 5 of them about O′. Now, conceiving the *convexity* of the group O in front and the *concavity* of group O′, we may place the two together so as to inclose a solid. Thus, placing A at 6, the three faces 5, 6, 1, will enclose a triedral, since the diedral included by 5 and 1 is the diedral of such a triedral. Then will vertex B fall at *c*, and a like triedral will be formed at that point, and so of all the other vertices. Hence we have a polyedron having for faces 12 equal regular pentagons.

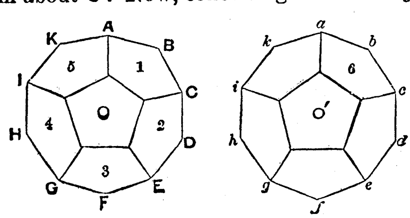

Fig. 402.

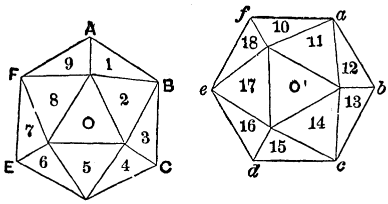

FIG. 403.

The Regular Icosaedron.—Taking 20 equal equilateral triangles, they can be grouped in two sets, as in the figure, in a manner altogether similar to the preceding case. The solid angles in this case are included by 5 facial angles whose sum is $3\frac{1}{3}$ right angles (?), which is a possible case (PART II., ***436***). As before, conceiving the *convexity* of group O in front, and the *concavity* of O', we can place them together by placing A at a, thus enclosing a solid angle with 5 faces, whence B will fall at b, etc. Thus we obtain a solid with 20 equal equilateral triangles for its faces.

That there can be no other regular polyedrons than these 5 is evident, since we can form no other convex solid angles by means of regular polygons. Thus, with equilateral triangles (the simplest polygon) we have formed solid angles with 3 faces (the least number possible), as in the tetraedron; with 4, as in the octaedron; and with 5, as in the icosaedron. Six such facial angles cannot enclose a solid angle, since their sum is four right angles (?), and much less any greater number. Again, with squares (the next most simple polygon) we have formed solid angles with 3 faces as in the hexaedron, and can form no other, for the same reason as above. With regular pentagons we can only enclose a triedral, as in the dodecaedron, for a like reason. With regular hexagons we cannot enclose a solid angle (?), and much less with any regular polygon of more than six sides.

SCH.—Models of the regular polyedrons are easily formed by cutting the following figures from cardboard, cutting half-way through the board in the dotted lines, and bringing the edges together as the forms will readily suggest.

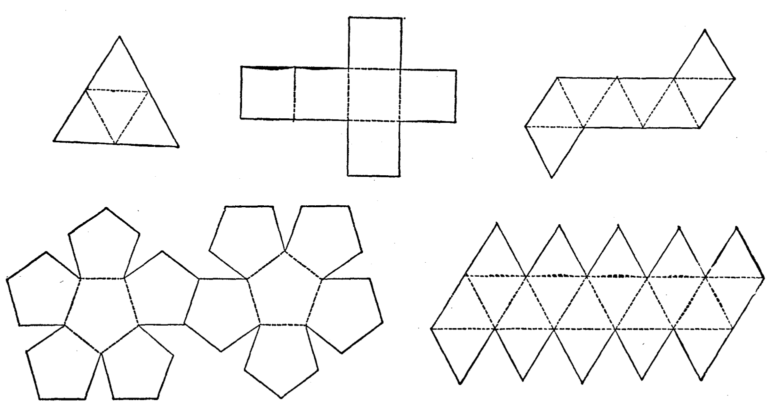

FIG. 404.

730. Any regular polyedron is inscriptible and circumscriptible by a sphere.

SUG'S.—From the centres of any two adjacent faces, as c and c', let fall perpendiculars upon the common edge, and they will meet it in the same point o (?). The plane of these lines will be perpendicular to this edge (?), and perpendiculars to these faces from their centres, as cS, c'S, will lie in this plane (?), and hence will intersect at a point equally distant from these faces.

In like manner c''S $= c'$S, and the point S can be shown to be equally distant from all of the faces, and is therefore the centre of the inscribed sphere.

Joining S with the vertices, we can readily show that S is also the centre of the circumscribed sphere.

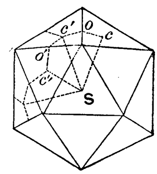

FIG. 405.

---

731. Show that a, being the edge of a regular tetraedron, its volume is $\frac{a^3\sqrt{2}}{12}$.

---

732. DEF.—***A Truncated Prism*** is one whose upper and lower bases are not parallel.

733. The volume of a truncated triangular prism is equal to the sum of the volumes of three pyramids whose common base is the lower base of the prism, and whose vertices are the angles of the upper base.

SUG'S.—Let bD, cD$'$, and aD$''$ be perpendicular to the lower base. Volume of b-ABC is $\frac{1}{3}$ bD $\times$ ABC. Volume a-bCc : volume b-ABC : : cbC : bBC : : cC : bB : : cD$'$: bD. $\therefore$ Volume a-bCa $= \frac{1}{3}c$D$' \times$ ABC. In a similar manner volume b-aAC $= \frac{1}{3}a$D$'' \times$ ABC.

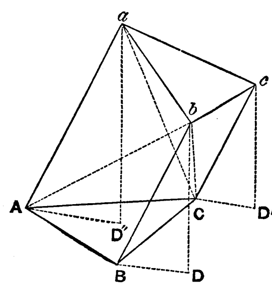

FIG. 406.

734. COR.—The volume of a prism, one of whose bases is a right section and the other an oblique section, is the product of the right section into the arithmetical mean of its edges.

SUG'S.—The volume of abc-ABC is as shown above ABC $\left(\frac{b\text{D} + c\text{D}' + a\text{D}''}{3}\right)$. But if ABC is a right section, bD $= b$B, cD$' = c$C, and aD$'' = a$A. Hence the volume is ABC $\left(\frac{b\text{B} + c\text{C} + a\text{A}}{3}\right)$.

735. The volume of any polyedron having for its bases any two polygons whatever, situated in parallel planes, and for lateral faces trapezoids, is the product of $\frac{1}{6}$ the distance between the bases into the sum of the two bases plus 4 times a section midway between the bases; or $V = \frac{H}{6}(B + B' + 4B'')$, in which H is the distance between the bases, B and B' the bases, and B'' a section midway between the bases.

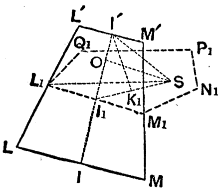

FIG. 407.

DEM.—Let $L_1 M_1 N_1 P_1 Q_1$ be the section of such a polyedron midway between its bases, and S any point in this section. Joining S with the vertices of the polyedron, we divide the solid into as many pyramids as it has faces. The volumes of the two which have B and B' for their bases are evidently $\frac{1}{6}H \times B$, and $\frac{1}{6}H \times B'$. It remains to find the volume of the others.

Let $LML'M'$ be a lateral face corresponding to L_1M_1 and SO a perpendicular from S upon this face. Draw $l'l$ through O perpendicular to LM, and consequently to $L'M'$. Take $l'K_1$ perpendicular to the plane section, whence $l'K_1 = \frac{1}{2}H$. Now the volume of the pyramid having $L'M'LM$ for its base and S for its vertex is $L_1M_1 \times 2l'l_1 \times \frac{1}{3}SO$. But $l'l_1 \times SO = Sl_1 \times l'K_1$ (?); whence the volume of this pyramid is $\frac{2}{3} L_1M_1 \times Sl_1 \times l'K_1 = \frac{2}{3} \times 2SL_1M_1 \times l'K_1 = \frac{1}{3} l'K_1 \times 4SL_1M_1 = \frac{1}{6}H \times 4SL_1M_1$. In like manner the volume of the pyramid having for its base the face in which M, N, is situated, can be shown to be $\frac{1}{6}H \times 4SM_1N_1$ and similarly of all the others. Whence the whole volume is $\frac{1}{6}H(B + B' + 4B'')$.

736. COR.—The proposition is equally true when some or all of the lateral faces are triangles; *i. e.*, when one base has more sides than the other.

SCH.—The preceding propositions are of much value in calculating earthwork.

---

737. If we cut a pyramid by a plane parallel to its base, a second pyramid is formed similar to the first.

---

738. Two triangular pyramids are similar whenever they have an equal diedral angle contained between faces, similar each to each, and similarly placed.

---

739. Two polyedrons composed of the same number of tetraedrons, similar each to each, and similarly disposed, are similar.

740. All regular polyedrons of the same number of faces are similar solids.

---

741. The intersection of the surfaces of two spheres is the circumference of a circle whose plane is perpendicular to the line which joins their centres.

---

742. Through any four points not in the same plane one sphere may be made to pass, and only one.

SUG'S.—The four points may be considered as the vertices of a tetraedron. Conceive in the triangular faces perpendiculars to the faces, from the intersections of lines drawn in these faces perpendicular to the sides at their middle points. These perpendiculars will meet at a common point (?), which is the centre of the circumscribed sphere (?).

[The student should show why *only* one sphere can be circumscribed.]

743. COR. 1.—The four perpendiculars erected at the centres of the faces of a tetraedron intersect at a common point.

744. COR. 2.—The six planes, perpendicular to the six edges of a tetraedron, intersect at the centre of the circumscribed sphere.

---

745. One sphere and only one may be inscribed in any tetraedron.

SUG.—Bisect the diedrals with planes.

---

746. The angle included by any two curves intersecting on the surface of a sphere, is equal to the angle included by the arcs of two great circles passing through the point of intersection, and whose planes produced include the tangents to the curves at their intersection.

SECTION II.

PROBLEMS IN SPECIAL OR ELEMENTARY GEOMETRY.

747. To bisect the angle formed by two lines whose intersection is inaccessible.

SUG.—M and N are points in the bisector.

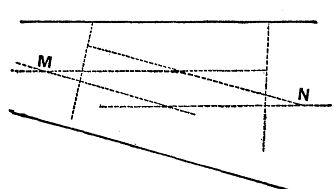

FIG. 408.

748. To pass a circumference through three points, not in the same straight line, when the radius is so long as to render the ordinary method impracticable.

SUG.—Let A, B, and C be the three points; then are M and N other points in the same circumference.

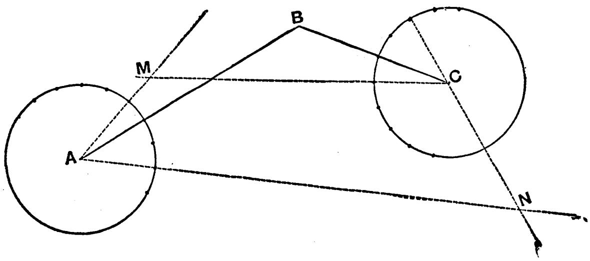

FIG. 409.

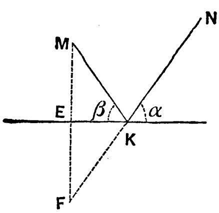

FIG. 410.

749. From two given points on the same side of a line given in position, to draw two lines which shall meet in that line and make equal angles with it.

SUG.—If α and β are equal, what is the relation of ME to EF?

---

750. To construct an isosceles triangle with a given base and vertical angle.

SUG.—See Prob. 4, p. 102.

---

751. To trisect a right angle.

SUG.—What is the value of an angle of an equilateral triangle?

---

752. Given the perpendicular of an equilateral triangle, to construct the triangle.

---

753. Given the diagonal of a square, to construct it.

---

754. To construct an isosceles triangle, so that the base shall be a given line, and the vertical angle a right angle.

755. Given the sum of the diagonal and a side of the square, to construct it.

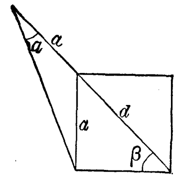

Fig. 411.

Sug.—What are the values of α and β respectively?

---

756. To construct a triangle when the altitude, the vertical angle, and one of the sides are given.

---

757. To construct a triangle when the sum of the three sides and the angles at the base are given.

Sug's.—MN being the sum of a, b, and c, what are the angles M and N as compared with the given angles α and β?

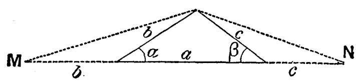

Fig. 412.

758. In a right angled triangle the perimeter, and the perpendicular from the right angle upon the hypotenuse being given, to construct the triangle.

Sug's.—DE is equal to the perimeter, DBE is an angle of 135°, and FE is the perpendicular on the hypotenuse. ABC is the required triangle. Let the student give the solution in full, and the proof.

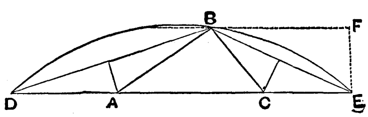

Fig. 413.

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759. From two given points on the same side of a given line, to draw two equal straight lines which shall meet in the same point of the line.

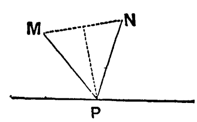

Fig. 414.

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760. To pass a circumference through two given points, which shall have its centre in a given line.

---

761. To construct a quadrilateral when three sides, one angle, and the sum of two other angles are given.

Sug's.—What is the fourth angle? In any case two sides and their included angle are known. There will be two cases according as the two angles whose sum is known are adjacent to each other or opposite. In the latter case we have to describe a segment on a diagonal, which will contain the fourth angle.

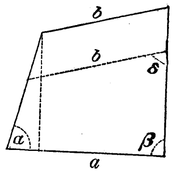

Fig. 415.

762. To construct a quadrilateral when three angles and two opposite sides are given.

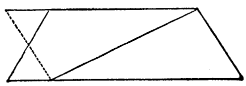

Fig. 416.

763. To bisect a trapezoid by a line drawn from one of its angles.

---

764. In a given circle, to inscribe a triangle equiangular with a given triangle.

Sug.—How does an angle at the centre compare with one inscribed in the same segment?

---

765. To describe three circles of equal diameters which shall touch each other.

---

766. In an equilateral triangle, to inscribe three equal circles which shall touch each other and the three sides of the triangle.

---

767. To describe a circle of given radius touching the two sides of a given angle.

Sug.—How far is the centre from each line?

---

768. To describe a circumference which shall be embraced between two parallels and pass through a given point within the parallels.

Sug.—In what line is the centre? How far from the given point?

---

769. To describe a circle with a given radius, which shall pass through a given point and be tangent to a given line.

---

770. To find in one side of a triangle the centre of a circle which shall touch the other two sides.

---

771. Through a given point on a circumference, and another

given point without, to describe a circle touching the given circumference.

SUG.—Consider in what two lines the centre must lie.

---

772. In the diameter of a circle produced, to determine the point from which a tangent drawn to the circumference shall be equal to the diameter.

SUG.—What is the relation between the radius, the required tangent, and the distance from the centre to the intersection of the produced diameter and the required tangent?

---

773. To describe a circle of given radius, touching two given circles.

---

774. In a given circle, to inscribe a right angle, one side of which is given.

---

775. In a given circle, to construct an inscribed triangle of given altitude and vertical angle.

---

776. To inscribe a square in a given right-angled isosceles triangle, one side being in the hypotenuse.

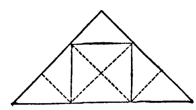

FIG. 417.

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777. To inscribe a square in a given quadrant of a circle, the vertex of an angle being at the centre.

---

778. To find the centre of a circle in which two given lines meeting in a point shall be a tangent and a chord.

---

779. To describe a circumference which shall pass through a given point and be tangent to a given line at a given point.

---

780. To bisect a quadrilateral by a line drawn from one of its angles.

SUG.—The demonstration is based upon the principle that triangles having equal bases and equal altitudes are equivalent.

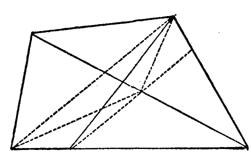

FIG. 418.

781. Through a given point situated between the sides of an angle, to draw a line terminating at the sides of the angle, and in such a manner as to be bisected at the point.

SUG.—Conceive the point as situated in the third side of a triangle of which the two given lines are the other two.

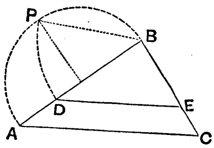

FIG. 419.

782. To draw a line parallel to the base of a triangle so as to divide the triangle into two equivalent parts.

SUG. $\overline{PB}^2 = \overline{DB}^2 = \frac{1}{2}\overline{AB}^2$. See (***344, 362***).

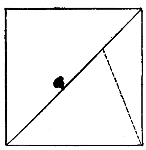
FIG. 420.

783. To construct a square when the difference between the diagonal and a side is given.

SUG.—Consider the angles.

784. To determine the point in the circumference of a circle from which chords drawn to two given points shall have a given ratio.

SUG.—Draw a chord dividing the chord joining the given points in the required ratio, and bisecting one of the subtended arcs.

785. To bisect a given triangle by a line drawn from one of its angles.

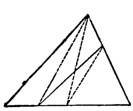
FIG. 421

786. To bisect a given triangle by a line drawn from a given point in one of its sides.

787. In the base of a triangle find the point from which lines extending to the sides, and parallel to them, will be equal.

788. To construct a parallelogram having the diagonals and one side given.

789. To construct a triangle when the three altitudes are given.

SUG.—What is the relation of the perpendiculars to the sides upon which they fall? If a triangle be formed with the perpendiculars as sides, how will it compare with the first triangle?

---

790. What is the area of the sector whose arc is 50°, and whose radius is 10 inches?

---

791. To construct a square equivalent to the sum, or to the difference of two given squares.

---

792. To divide a given straight line in the ratio of the areas of two given squares.

---

793. To construct a triangle, when the altitude, the line bisecting the vertical angle, and the line from the vertex to the middle of the base are given.

SUG.—The centre of the circle circumscribing the required triangle is in the perpendicular to the base at its middle point, and also in a line which makes the same angle with the bisecting line that the bisector does with the perpendicular.

Show that the bisector always lies between the perpendicular and the medial line.

---

794. Through a given point, draw a line such that the parts of it, between the given point and perpendiculars let fall on it from two other given points, shall be equal.

What would be the result, if the first point were in the straight line joining the other two?

---

795. From a point without two given lines, to draw a line such that the part intercepted between the given lines shall be equal to the part between the given point and the nearest line.

SUG.—Produce the lines till they meet, if necessary. Draw a line through the given point parallel to one of the lines, and produce it till it meets the other.

---

796. Given one angle, a side adjacent to it, and the difference of the other two sides, to construct the triangle.

QUERIES.—How if $b > a$? How if B is obtuse?

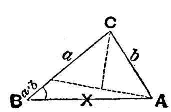

FIG. 422.

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797. To pass a circumference through two given points, having its centre in a given line.

798. To draw a line parallel to a given line and tangent to a given circumference.

Sug.—Draw a diameter perpendicular to the given line.

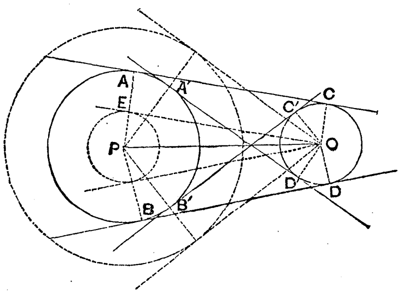

Fig. 423.

799. To draw a common tangent to two given circles.

Sug.—1st Method.—There are two sets of tangents, AC, BD, and A′D′, B′C′. For the first, observe that if $PE = AP - OC$, OE is parallel to AC, etc.

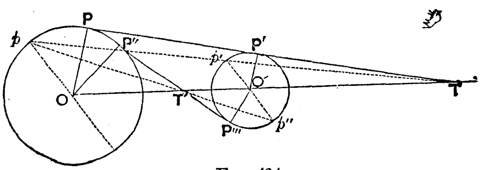

Fig. 424.

2d Method. pO, $p'O'$ being parallel to each other, $pp'T$ gives the intersection of the tangent with the line passing through the centres, since

$$pO : p'O' :: OT : O'T, \text{ or } pO - p'O' : p'O' :: OO' : O'T.$$

$$\text{Also, } PO - P'O' : P'O' :: OO' : O'T.$$

Hence O′T is constant for all positions of the parallel radii. Prove that if the parallel radii are on different sides of the line joining the centres, T′ is the point where the internal tangent cuts OO′.

Queries.—How many tangents can be drawn—1st. When the circles are external one to the other; 2d. When they are tangent externally; 3d. When they intersect each other; 4th. When tangent internally; 5th. When one lies within the other?

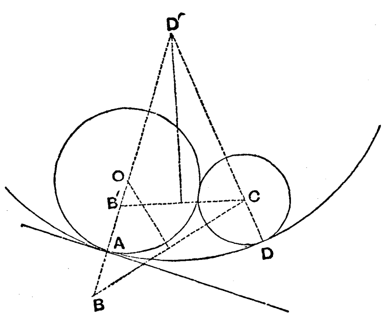

Fig. 425.

800. To describe a circle tangent to a given circumference and also to a given line at a given point.

Sug's.—There may be two cases—1st. When the given circle is exterior to the one sought; and 2d. When it is interior. In either case the centre of the required circle is in the perpendicular AO′. In the former case, O, the centre of the required circle, is at $r + r'$ from C; and in the latter O′ is at $r - r'$ from C. $AO = r$, and $CD = AB = AB' = r'$.

801. To construct a trapezoid when the four sides are given.

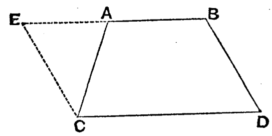

FIG. 426.

SUG.—Knowing the difference between the two parallel sides, we may construct the triangle AEC, and hence the trapezoid.

802. On a given line, to construct a polygon similar to a given polygon.

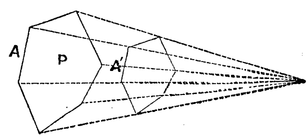

FIG. 427.

SUG'S.—One method may be learned from (***90***). Ex. 8, page 152, furnishes another method. The following is an elegant method: To construct on A′ homologous with A, a polygon similar to P. Place A′ parallel to A, and the figure will suggest the construction.

803. To pass a plane through a given line and tangent to a given sphere.

SUG'S.—Pass a plane through the centre of the sphere and perpendicular to the given line. Through the point of intersection and in this secant plane draw tangents to the great circle in which the secant plane intersects the surface of the sphere. The points of tangency will be the points of tangency of the required planes (?), of which there are thus seen to be two.

804. DEF.—***A Tangent Plane*** to a cylindrical or conical surface is a plane which contains an element of the surface, but does not cut the surface. The element which is common to the surface and the plane is called the *Element of Contact.*

805. To pass a plane through a given point and tangent to a given cylinder of revolution.

SUG'S.—1st. When the point is in the surface of the cylinder. Through the point draw an element of the cylinder, by passing a line parallel to the axis, or to any given element. Through the same point pass a plane perpendicular to this element, making a right section (a circle). To this circle draw a tangent. The plane of the element and tangent is the tangent plane required. [The student should prove that any point in the plane affirmed to be tangent, not in the element passing through the given point, is without the cylinder.]

2d. When the given point is without the cylinder. Pass a plane through the given point perpendicular to the axis of the cylinder, thus making a right section of the cylinder (a circle). In this secant plane draw tangents to the section. Through the points of contact of these tangents draw elements of the cylinder. These elements are the elements of contact of the tangent planes. Hence planes passing through them and the given point are the tan-

gent planes required. [The student should remember that this is but an *outline*, and be careful to fill it up, giving the proof.]

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806. To pass a plane through a given point and tangent to a conical surface of revolution.

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807. To find, with the compasses and ruler, the radius of a material sphere whose centre is inaccessible.

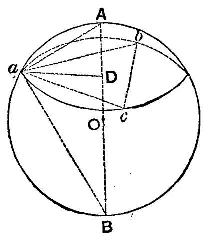

FIG. 428.

SUG'S.—With one point of the compasses at any point in the surface, as A, trace a circle of the sphere, as *acb*. The chord A*a* is measured by the distance between the compass points. In like manner measure three other chords, as *ac*, *ab*, and *bc*. Draw a plain triangle having these chords for its sides, and circumscribe a circle about it. Thus *a*D is found. Knowing *a*A, and *a*D, and remembering that A*a*B is right angled at *a*, the triangle A*a*B can be drawn in a plane (?), whence AO becomes known.

SECTION III.

APPLICATIONS OF ALGEBRA TO GEOMETRY.

808. The mathematical method which is called technically *Applications of Algebra to Geometry* consists in finding, by means of equations, the numerical values of the unknown parts of a geometrical figure, when a sufficient number of the parts are given numerically.

809. By reference to the COMPLETE SCHOOL ALGEBRA, page 238, it will be seen that the algebraic solution of a problem consists of two parts: 1st. The *Statement*, which is the expressing by one or more equations of the conditions of the problem, *i. e.*, the relations between the known and unknown quantities (parts of the figure) to be compared; and 2d. The *Solution* of these equations, so as to find the values of the unknown quantities in known ones.

810. In applying the equation for the solution of such problems as are now proposed, we have to depend upon our previously acquired knowledge of the properties of geometrical figures for the relations between the known and unknown quantities, which will enable us to form the necessary equations, *i. e.*, to make the *State-*

ment. The resolution of the equations thus arising is effected in the ordinary ways. [See NOTE, page 239 of THE COMPLETE SCHOOL ALGEBRA.]

811. The details of this method will be most readily obtained from a careful study of examples.

EXAMPLES.

812. In a right angled triangle, given the hypotenuse and the sum of the other two sides, to find these sides separately.

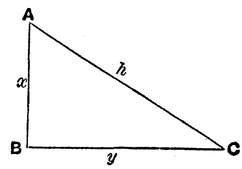

FIG. 429.

SOLUTION.—Let ABC be a triangle, right angled at B. Let the *known* hypotenuse be h, the *unknown* base, y; the *unknown* altitude, x; and the *known* sum of the base and altitude, s.

We have here two unknown quantities, and hence must have two equations, in order to find their values. One of these equations is furnished directly by the statement of the problem, which says that the sum of the base and perpendicular is to be given. Hence—

Equation 1 is $$x + y = s.$$

A second relation between x and y and the known quantity h is furnished by the relation given in PART II. (***346***). Whence—

Equation 2 is $$x^2 + y^2 = h^2.$$

Solving these equations we find—

$$y = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}, \text{ and } x = \tfrac{1}{2}s \mp \tfrac{1}{2}\sqrt{2h^2 - s^2}.$$

If $h = 10$ and $s = 14$, we find $x = 6$, and $y = 8$; or $x = 8$, and $y = 6$.

GEOMETRICAL SOLUTION—It is exceedingly interesting and instructive to compare the algebraic solution of such problems with their geometrical solution, when the problem can be solved in both ways. The geometrical solution of this problem is as follows:

Take DC $= s$, the sum of the two sides, and make ODC $= 45°$. From C as a centre, with a radius h, the hypotenuse, describe an arc cutting DO, as in A and A′. Draw AC and the perpendicular AB, also A′C and the perpendicular A′B′. Both the triangles ABC and A′B′C fulfil the conditions. For AB = DB (?), whence AB + BC $= s$, and AC $= h$, by construction. So, also, A′B′= DB′ (?), whence A′B′ + B′C $= s$, and A′C $= h$, by construction.

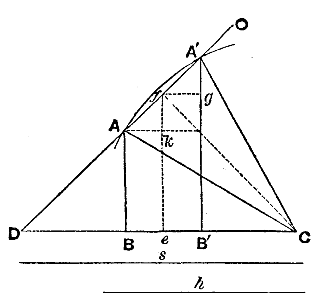

FIG. 430.

COMPARISONS OF THESE SOLUTIONS.—1st. We find in the *algebraic* solution, that, in

general, y may have two values—viz., $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, and $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$; and that when $y = \frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, $x = \frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$; but, when $y = \frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$, $x = \frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$. Correspondingly, we find in the *geometrical* solution that the base (y) may have two values—viz., BC, and B′C; and that when the base is BC, the altitude (x) is AB; but, when the base is B′C, the altitude is A′B′.

2d. From the algebraic solution, we observe that the base $y = \frac{1}{2}s \pm \frac{1}{2}\sqrt{2h^2 - s^2}$, may be considered as made up of two parts—viz., a rational part, $\frac{1}{2}s$, and a radical part, $\frac{1}{2}\sqrt{2h^2 - s^2}$; and that the altitude, $x = \frac{1}{2}s \mp \frac{1}{2}\sqrt{2h^2 - s^2}$, is made up of the *same* parts, only observing that, if the base is considered as the *sum* of these parts—viz., $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, the altitude is their *difference*—viz., $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$. If, however, the base is $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$, the altitude is $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$. Now, we can discover exactly the same things *geometrically*, and can show exactly what is the geometrical meaning of each of the parts of the values of y and x. To do this, draw Cf* bisecting AA′; let fall the perpendicular fe, and draw Ak and fg parallel to DC. Cf is perpendicular to DO (?), and hence equal to Df (?). Also, De = eC = fe = $\frac{1}{2}s$ (?). From the right angled isosceles triangle DfC, $f\text{C} = \frac{1}{\sqrt{2}}\text{DC} = \frac{s}{\sqrt{2}}$ (?). Hence, from AfC, $\text{A}f = \sqrt{\overline{\text{AC}}^2 - \overline{f\text{C}}^2} = \sqrt{h^2 - \frac{1}{2}s^2} = \frac{1}{\sqrt{2}}\sqrt{2h^2 - s^2}$. Again, from the right angled isosceles triangle Akf, we have $\text{A}k = \frac{1}{\sqrt{2}}\text{A}f\,(?) = \frac{1}{2}\sqrt{2h^2 - s^2}$. But A$k$ = fk = fg = A′g = Be = eB′. Hence we see that the rational part of the value of y ($\frac{1}{2}s$) is eC, and that the radical part ($\frac{1}{2}\sqrt{2h^2 - s^2}$) is B$e$, or eB′. In the triangle ABC the *sum* of these parts is the base; and in the triangle A′B′C, their *difference* is the base. In like manner fe represents the rational part of the value of x, and fk = A′g, the radical part.

3d. From the algebraic solution we see that if $s^2 = 2h^2$, $y = \frac{1}{2}s$, and $x = \frac{1}{2}s$. The same thing is seen in the geometrical solution, for if $s^2 = 2h^2$, $h = \frac{1}{\sqrt{2}}s$, or fC; whence the arc struck from C as a centre, with h as a radius, would be tangent to DO, instead of intersecting it in *two* points. Again, if $s^2 > 2h^2$, the quantity under the radical sign is negative, and the radical becomes *imaginary*. This means, that *no triangle* can be formed under these circumstances. This case appears in the geometrical solution also, for then $h < \frac{1}{\sqrt{2}}s$, or less than fC, and consequently the arc struck from C as a centre, with radius h, will not touch DO, and we get no triangle.

---

* This part of the construction should not be allowed on the figure till it is wanted—*i.e.*, till this stage of the discussion.

813. SCH.—This problem is discussed thus at length as an illustration of what *may* be done by such methods. Of course, all problems are not equally fruitful; but the student should not rest satisfied with a mere determination of the values of the unknown parts in known terms, when anything farther is revealed either by the process or result of the algebraic solution. Especially should he desire to become expert in seeing what geometrical relations are indicated by the *form* of the answer obtained.

---

814. Given the lengths of the medial lines from the acute angles of a right angled triangle, to determine the triangle, *i. e.*, to find the base and perpendicular.

SUG'S.—Let AD $= a$, CE $= b$, AB $= 2x$, and CB $= 2y$; then $4x^2 + y^2 = a^2$, and $4y^2 + x^2 = b^2$ (?). $\therefore 2x =$ AB $= \sqrt{\dfrac{4a^2 - b^2}{15}}$, and $2y =$ CB $= \sqrt{\dfrac{4b^2 - a^2}{15}}$.

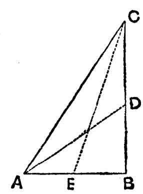

FIG. 431.

The *form* of these results indicates that CB sustains the same relation to CE and AD that AB does to AD and CE—a fact which is evident from the nature of the case.

Again, if $4a^2 < b^2$, $2a$ is imaginary; and if $4b^2 < a^2$, $2y$ is imaginary. In either case the triangle cannot exist. So also if $4a^2 = b^2$ $2x = 0$; and if $4b^2 = a^2$, $2y = 0$, and there can be no triangle. This may be seen from the figure by conceiving AB, for example, to diminish. As A approaches B, AD approaches equality with DB, and CE with CB. Hence the *limit* is AD $= \frac{1}{2}$CE.

Thus we see *that either medial line must be more than half the other*,—a proposition which is proved by this solution.

---

815. The hypotenuse and radius of the inscribed circle of a right angled triangle being given, to determine the triangle.

Results.—Calling the hypotenuse h, the radius r, the base x, and the perpendicular y, we have, $x = \dfrac{2r + h \pm \sqrt{h^2 - 4hr - 4r^2}}{2}$, and

$$y = \frac{2r + h \mp \sqrt{h^2 - 4hr - 4r^2}}{2}.$$

The results being the same in other respects, the double sign before the radical indicates that the base and perpendicular are interchangeable—a fact which is evident from the nature of the case.

If the radical is 0, *i. e.*, if $h^2 - 4hr - 4r^2 = 0$, $x = r + \frac{1}{2}h$, and $y = r + \frac{1}{2}h$, and the base and perpendicular are equal. Let the student show the same thing geometrically (from a figure).

Also, if $h^2 - 4hr - 4r^2 = 0$, $h = 2r\,(1 \pm \sqrt{2})$. In this result the negative

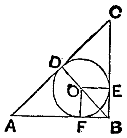

FIG. 432.

sign is to be rejected, since it would make h negative, as $\sqrt{2} > 1$.

The value $h = 2r\,(1 + \sqrt{2})$ is readily seen from the figure when AB = CB. Thus AC = 2DB = 2 (DO + OB) = $2\,(r + r\,\sqrt{2}) = 2r\,(1 + \sqrt{2})$ (?).

---

816. A tree of known height standing perpendicular on a horizontal plane, breaks so that its top strikes the ground at a given distance from the foot, while the other end hangs on the stump. How high is the stump? That is, given the base and the sum of the perpendicular and hypotenuse of a right angled triangle, to determine the perpendicular.

Result.—Let a be the height of the tree, b the distance from the foot to the point where the top strikes, and x the height of the stump; then $x = \frac{a^2 - b^2}{2a}$.

Since $\frac{a^2 - b^2}{2a} = \frac{1}{2}a - \frac{b^2}{2a}$, $\frac{b^2}{2a}$ is the distance below the middle, at which the tree breaks.

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817. In a rectangle, knowing the diagonal and perimeter, to find the sides.

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818. Knowing the base, b, and altitude, a, of any triangle, to find the side of the inscribed square, x.

Result, $x = \frac{ab}{a + b}$.

---

819. In an equilateral triangle, given the lengths, a, b, c, of the three perpendiculars from a point within upon the sides, to determine the sides.

SUG'S.—Find an expression for the altitude in terms of the sides; and then get two expressions for the area of the whole triangle. Equate these.

Result, each side $= \frac{2\,(a + b + c)}{\sqrt{3}}$.

---

820. In a right angled triangle whose hypotenuse is h, and difference between the base and perpendicular d, to find these sides.

Results, $x = \frac{-\,d \pm \sqrt{4h^2 - d^2}}{2}$, $x + d = \frac{d \mp \sqrt{4h^2 - d^2}}{2}$.

QUERIES.—What is the geometrical significance of the fact that the results are the same except as regards the signs? In the first result why must the negative value of the radical be rejected; and in the second the positive?

---

821. In an equilateral triangle given the lines a, b, c, drawn from a point within or without, to find the sides.

Result.—Each side =

$$\left\{\frac{a^2 + b^2 + c^2 \pm \sqrt{6\,(a^2b^2 + b^2c^2 + c^2a^2) - 3\,(a^4 + b^4 + c^4)}}{2}\right\}^{\frac{1}{2}}$$

The radical is + when the point is within, and − when it is without.

---

822. The perimeter of a right angled triangle and the perpendicular from the right angle upon the hypotenuse being given, to determine the triangle.

SUG'S.—Let s be the perimeter, p the perpendicular upon the hypotenuse, and $x + y$, $x - y$ the two sides about the right angle. Then the hypotenuse $= s - 2x$, and we readily form the two equations $p\ (s - 2x) = x^2 - y^2$, and $(x + y)^2 + (x - y)^2 = (s - 2x)^2$ (?). Hence $x = \frac{s(s + 2p)}{4(s + p)}$, and this value substituted in either equation will give y.

---

823. The base of a plane triangle is b and its altitude a, required the distance from the vertex at which a parallel to the base must cut the altitude in order to bisect the triangle.

$$\textit{Result,}\ \frac{a}{1 + \sqrt{2}}.$$

QUERY.—What does the fact that b does not appear in the result show?

---

824. Having given the area of a rectangle inscribed in a triangle, can the triangle be determined? Can it, if the rectangle is a square? If the rectangle is a square and the triangle right angled? If the rectangle is a square and the triangle equilateral?

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825. The sides of a triangle being a, b, c, to find the perpendicular upon c from the opposite angle.

$$\textit{Result,}\ p = \frac{1}{2c}\sqrt{2c^2\,(a^2 + b^2) + 2a^2b^2 - a^4 - b^4 - c^4}.$$

SUG'S.—Observe that a and b are similarly involved in the result, but c is differently involved from either. This is evidently as it should be, since a and b are the sides about the angle from which p is let fall; and are thus similarly related to p. But c, the side on which p falls, is differently related to p from

either of the others. The student should be able to write the value of the perpendiculars upon each of the other sides, from this one. Thus, that on a is

$$\frac{1}{2a}\sqrt{2a^2\,(c^2+b^2)+2c^2b^2-a^4-b^4-c^4}.$$

---

826. The sides of a triangle are a, b, c, to find the side of an inscribed square one of whose sides falls in c.

SUG'S.—The altitude may be found from the preceding, hence may be assumed as known. Call it p. Then the side of the required square is $\frac{cp}{c+p}$. What is the side of the square standing on a? On b?

QUERY.—Will the square be the same on whichever side it stands? Observe that though the values here found are apparently different, they *may* not be so really, since p is different in each case. But let the student decide.

---

827. Having the area of a rectangle inscribed in a given triangle and standing on a specified side, to determine the sides of the rectangle.

Result, b being the base on which the rectangle stands, p the altitude from this base, and s the given area, we have for the sides

$$x=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-\frac{sb}{p}},\ \text{and}\ y=\frac{p}{2}\mp\sqrt{\frac{p^2}{4}-\frac{sp}{b}}.$$

SUG'S.—The $\pm$ and $\mp$ signs indicate that, in general, there can be two equal rectangles inscribed standing on the same base. The student will do well to illustrate it with definite numerical values, as $p=10$, $b=6$, $s=10$.

Again, $\frac{b^2}{4}$ must be greater than $\frac{sb}{p}$, and $\frac{p^2}{4}>\frac{sp}{b}$, *i. e.*, s must be less than $\frac{1}{4}pb$.

That is, the greatest rectangle is half the area of the triangle, since $\frac{1}{2}pb$ is the area of the triangle.

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828. The Algebraic solution of a problem often enables us to effect a geometrical construction. We will give a few examples.

Through a given point within a circle, to draw a chord of a given length.

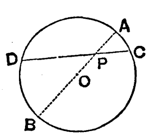

FIG. 433.

SOLUTION.—Let s be the length of the required chord, and P the given point. Since P is a known point, call AP $=a$, PB $=b$, AB being the diameter through P. Let CD represent the required chord, and calling CP, x, PD $=s-x$. Then $sx-x^2=ab$; whence $x=\frac{1}{2}s\pm\sqrt{\frac{1}{4}s-ab}$.

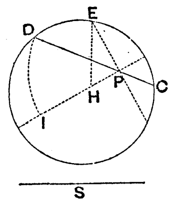

FIG. 434.

To effect the geometrical construction, let s be the given chord, and P the point in the given circle. Draw the diameter through P, and erect PE perpendicular to it. Make EH $= \frac{1}{2}s$; then since $\overline{PE}^2 = ab$, PH $= \sqrt{\frac{1}{4}s^2 - ab}$. Now take PI $= \frac{1}{2}s$, and from P as a centre, with a radius PI $= \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 - ab}$, strike the arc DI intersecting the circumference. DPC is the chord required.

From the radical $\sqrt{\frac{1}{4}s^2 - ab}$ we see that, if $ab > \frac{1}{4}s^2$, x is imaginary, as we say in algebra. In such a case the problem is geometrically impossible, as will appear from the construction, for then PE is greater than EH, which makes HP, the representative of $\sqrt{\frac{1}{4}s^2 - ab}$, impossible. If $\frac{1}{4}s^2 = ab$, x has but one value, and the segments are equal.

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829. To find a point in a tangent to a circle from which, if a secant be drawn to the extremity of the diameter passing through the point of tangency, the external segment shall have a given length.

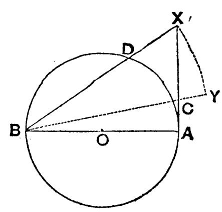

FIG. 435.

SOLUTION.—Let AB $= d$ be the diameter of the given circle, DX $= a$ the external segment of the required secant, and the whole secant BX $= x$. Then

$$x^2 - ax = d^2, \text{ and } x = \tfrac{1}{2}a \pm \sqrt{d^2 + \tfrac{1}{4}a^2}.$$

To effect the geometrical construction, construct the radical by taking AC $= \frac{1}{2}a$; whence BC $= \sqrt{d^2 + \frac{1}{4}a^2}$. Now make CY $= \frac{1}{2}a$, and with B as a centre, and BY as a radius, strike an arc cutting the tangent, as in X. Then is

$$\text{BX} = x = \tfrac{1}{2}a + \sqrt{d^2 + \tfrac{1}{4}a^2}.$$

The negative value of the radical is inapplicable in this elementary, geometrical sense, since as $\sqrt{d^2 + \frac{1}{4}a^2} > \frac{1}{2}a$, this would make x a negative quantity. Again we see that no real value of a can render x imaginary.

We can observe the same things from the geometrical construction. Thus, if the negative value of the radical were taken, x would be *numerically* less than BC, by $\frac{1}{2}a$, or AC. But BC $-$ AC $<$ BA. Hence an arc struck from B with the required radius would not cut the tangent. We see also that a may have any value between 0 and ∞.

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830. Given the hypotenuse and area of a right angled triangle, to construct the triangle.

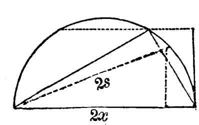

FIG. 436.

SUG'S.—Let h be the hypotenuse, s^2 the area, and x the perpendicular from the right angle upon the hypotenuse. Then $hx = 2s^2$, or $\frac{1}{2}h : s :: s : x$, and $h : 2s :: 2s : 2x$.

The figure will suggest the construction.

831. Through a point between two lines which intersect, to draw a line which shall cut off a triangle of given area.

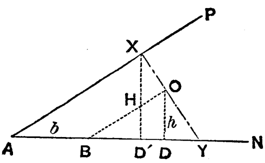

FIG. 437.

SUG'S.—Let AY $= x$, and the required area $= s^2$. We have $h : \mathrm{H} :: x - b : x$. $\therefore \mathrm{H} = \frac{hx}{x-b}$. And $\mathrm{H}x = 2s^2$. $\therefore \mathrm{H} = \frac{2s^2}{x}$. Thus

$$x = \frac{s^2}{h} \pm \sqrt{\frac{s^2}{h}\left(\frac{s^2}{h} - 2b\right)}.$$

To construct this, find $c = \frac{s^2}{h}$, *i. e.*, construct a third proportional to h and s. Then construct $\sqrt{c\,(c - 2b)}$, *i. e.*, find a mean proportional between c and $c - 2b$; let this be m. Whence $x = c \pm m$. In general, there may be two solutions, if any, since there are two values of x. [This should also be observed from the figure.] But if $2b > \frac{s^2}{h}$ there is no solution. If $\frac{s^2}{h} = 2b$, there is but one solution. In the latter case where is the given point O? What is the geometrical difficulty when $2b > \frac{s^2}{h}$? Can m be numerically greater than c?

832. To construct the four forms of the affected or complete quadratic equation, viz., (1.) $x^2 + px - q = o$, (2.) $x^2 - px - q = o$, (3.) $x^2 - px + q = o$, (4.) $x^2 + px + q = o$, without solving the equations.

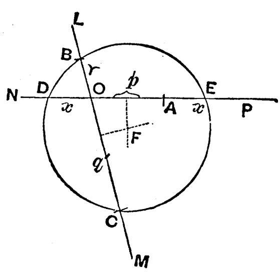

FIG. 438.

FIRST FORM. $x^2 + px - q = o$.—Draw any two lines as LM, NP, intersecting in some point O. Resolve q of the equation into two factors, as r and q', so that we have $x^2 + px - r \times q' = o$. Take OA $= p$, OB $= r$, OC $= q'$. Bisect CB and AO by perpendiculars, and from their intersection F as a centre, with a radius FB, draw a circle. Then DO, or AE, is x, the positive root. For $x\,(x + p) = rq'$, or $x^2 + px - rq' = o$. The negative root is OE. Thus, let OE $= (-\,x)$. Then DO $=$ AE $= (-\,x - p)$. Hence $(-\,x)(-\,x - p) = x^2 + px = rq'$, or $x^2 + px - rq' = o$.

This construction is evidently always possible irrespective of the relative magnitudes of p, r, q'; a fact which agrees with the statement in algebra that this form always has *real roots.*

SECOND FORM. $x^2 - px - rq' = o$.—The construction is the same as for the first form; only, in this case OE is the positive, and DO the negative root. Thus for OE $= x$ (positive), we have DO $\times$ OE $= (x - p)\,x = rq'$, or $x^2 - px -$

$rq' = o$. For DO $= (-x)$, we have DO $\times$ OE $=$ DO (OA $+$ AE) $=$ DO (OA $+$ DO) $= (-x)(p-x) = rq'$, or $x^2 - px - rq' = o$.

Observe that in the first case the negative root is numerically greater than the positive; while it is the reverse in this form. This agrees with the conclusions of algebra (See COMPLETE SCHOOL ALGEBRA, ***104***).

THIRD FORM. $x^2 - px + rq' = o$.—Draw any two lines, as OM, OP, meeting at O. Take OA $= p$, OB $= r$ or q', and OC $= q'$ or r. Erect perpendiculars at the middle points of OA, and BC; and from their intersection F as a centre, with a radius FB, strike a circumference. Then OE and OD are the values of x. For OE $= x$, OE $\times$ OD $=$ OE $\times$ EA $=$ OE (OA $-$ OE) $= x(p-x) = rq'$, or $x^2 - px + rq' = o$. For OD $= x$, OD $\times$ OE $=$ OD (OA $-$ AE) $=$ OD (OA $-$ OD) $= x(p-x) = rq'$, or $x^2 - px + rq' = o$.

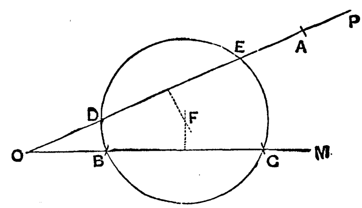

FIG. 439.

Observe that the former value of x is greater than the latter, but that neither is negative.

So also, we may readily see that the roots may become equal, and also, imaginary. Thus if the circle were tangent to OA, the roots would be equal, and if it did not touch OA they would *both* be imaginary. (See Algebra, as above.)

FOURTH FORM. $x^2 + px + rq' = o$.—The construction is the same as the last, only both values of x are negative. Thus, $(-x)[p-(-x)] = (-x)(p+x) = rq'$, $-px - x^2 - rq' = o$, or $x^2 + px + rq' = o$.

SCH.—Thus we see that we can construct any equation of the second degree containing but one unknown quantity, which has real roots. Hence, if the algebraic solution of a geometrical problem requires only the resolution of such an equation, the algebraic solution will lead to the geometrical construction.

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833. We have now given sufficient illustrations of this most interesting and important subject, so that the student should have caught the spirit of this method of using algebra to subserve the purposes of geometrical investigation. We shall simply append a list of problems, upon which the student can put in exercise both his algebraic and geometric knowledge. But we cannot refrain from repeating the advice, that the learner should not rest satisfied with the mere algebraic resolution of the problem. He should be ambitious to trace, *as fully as possible*, the wonderful relations which exist between the abstract operations of algebra, and the more concrete relations of geometry.

EXAMPLES.

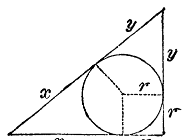

FIG. 440.

834. Given the perimeter of a right angled triangle and the radius of the inscribed circle, to determine the triangle.

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835. Given the hypotenuse of a right angled triangle and the side of the inscribed square, to determine the triangle.

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836. In a right angled triangle, given the radius of the inscribed circle, and the side of the inscribed square, the right angle of the triangle constituting one angle of the square, to determine the triangle.

SUG'S.—Letting x and y be the sides, z the hypotenuse, r the radius of the inscribed circle, and s the side of the inscribed square, we have $s = \dfrac{xy}{x+y}$, $xy = r(x + y + z)$, and $x + y = z + 2r$. Whence $z = 2r\left(\dfrac{s}{2r} - \dfrac{r}{s}\right)$, etc

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837. In any triangle whose sides are a, b, c, to find the radius of the inscribed circle.

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838. Show that the area of a regular dodecagon inscribed in a circle whose radius is 1, is 3.

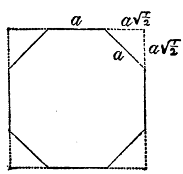

FIG. 441.

839. Find the area of a regular octagon whose side is a.

Result, $2(\sqrt{2} + 1)\, a^2$.

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840. Find the radii of three equal circles described in a given circle, tangent to the given circle and to each other.

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841. The space between three equal circles tangent to each other is a; what is the radius?

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842. In a triangle, given the ratio of two sides, and the segments of the third side made by a perpendicular let fall from the angle opposite.

843. In a triangle, given the base and altitude, and the ratio of the other sides, to determine the triangle.

844. Given the base, the medial line, and the sum of the other sides of a triangle, to determine the triangle.

845. To determine a right angled triangle, knowing the perimeter and area.

SUG'S. $x^2 + y^2 = z^2$, $x + y + z = 2p$, and $xy = 2s^2$, give $y + x = 2p - z$, $x^2 + 2xy + y^2 = 4p^2 - 4pz + z^2$, $z^2 + 4s^2 = 4p^2 - 4pz + z^2$; whence $z = \frac{p^2 - s^2}{p}$. Now use $y + x = 2p - z = \frac{p^2 + s^2}{p}$, and $xy = 2s^2$.

846. To determine a right angled triangle, knowing the perimeter, and the sum of the hypotenuse, and the perpendicular upon the hypotenuse from the right angle.

SUG'S. $x^2 + y^2 = z^2$, $x + y + z = 2p$, $z + u = a$, $xy = zu$. Then $x^2 + 2xy + y^2 = 4p^2 - 4pz + z^2$; whence $2xy = 4p^2 - 4pz$, and hence $2z(a - z) = 4p - 4pz$, etc.

847. The volume, the altitude, and a side of one of the bases of the frustum of a square pyramid being known, to determine a side of the other base.

848. To determine a right angled triangle, knowing the perimeter, and the perpendicular let fall from the right angle upon the hypotenuse.

849. To determine a triangle, knowing the base, the altitude, and the difference of the other sides.

850. To determine a triangle, knowing the base, the altitude, and the rectangle of the other sides.

851. To determine a right angled triangle, knowing the hypotenuse and the difference between the lines drawn from the acute angles to the centre of the inscribed circle.

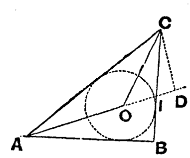

FIG. 442.

SUG'S.—Let fall CD a perpendicular upon AO produced. Now, since the the angles BAC and ACB are bisected, and COD = OAC + OCA, and ICD = IAB, they being complements of the equal angles CID, IAB, we have, COD = OCD, and CD = OD = $\sqrt{\frac{1}{2}}$ CO. Hence, putting AC $= h$, CO $= x$, and AO $= x + d$, we have

$$(x + d + \sqrt{\tfrac{1}{2}}\, x)^2 + (\sqrt{\tfrac{1}{2}}\, x)^2 = h^2.$$ From this x is readily found. The student should then be able to complete the solution.

852. Given two sides of a triangle and the bisector of their included angle, to determine the triangle.

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853. Given the three medial lines, to determine a triangle.

---

854. Given the three sides of a triangle, to determine the radius of the circumscribed circle.

---

855. Four equal balls whose radius is r are placed on a plane so that each is tangent to the other three, thus forming a pyramid; what is its altitude?

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856. Given the base of a triangle, the bisector of the opposite angle, and the radius of the circumscribing circle, to determine the triangle.

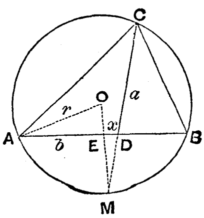

FIG. 443.

SUG'S.—First to find ED $= x$. Since EM $= r - \sqrt{r^2 - b^2}$, it may be considered known and put equal to c. We then have DM $= \sqrt{c^2 + x^2}$; and also, DM $\times\, a =$ AD $\times$ DB $= b^2 - x^2$, or DM $= \dfrac{b^2 - x^2}{a}$.

Whence $\sqrt{c^2 + x^2} = \dfrac{b^2 - x^2}{a}$, and x is readily found. Calling ED $= s$, the student will have no difficulty in proceeding with the solution.

CHAPTER II.

*INTRODUCTION TO MODERN GEOMETRY.**

SECTION I.

OF LOCI.

857. The term ***Locus*** †, as used in geometry, is nearly synonymous with *geometrical figure*, yet having a latitude in its use which the other does not possess. *The locus of a point* is the line (geo-

* With strict propriety only the latter sections of this chapter belong to the *Modern Geometry*, technically so called. But, as the entire chapter is composed of matter which has not hitherto found place in our common text-books, and the relative importance of which is becoming more fully appreciated in modern times, the author has ventured to embrace the whole under this title.

† The word *Locus* is the Latin for *place*.

metrical figure) generated by the motion of the point according to some given law.

In the same manner, a surface is conceived as the locus of a line moving in some determinate manner.

Ill's.—The locus of a point in a plane, which point is always equidistant from the extremities of a given right line, is a straight line perpendicular to the given line at its middle point. Thus, suppose AB a fixed line, and *the locus of a point* equidistant from its extremities is required; that point may be anywhere in a perpendicular to AB at its middle point, *and cannot be anywhere else* in this plane.

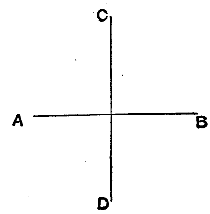

Fig. 444.

This perpendicular is the locus (place) of a point subject to the given law.

Again, a boy on the green is required to keep at just 20 feet from a certain stake; where may he be found? *i. e.*, what is his locus (place)? Evidently, the circumference of a circle whose radius is 20 feet. Thus, the locus of a point in a plane, equidistant from a given point, is the circumference of a circle. This is the *place* of such a point.

What is the locus in space of a point equidistant from a given point?

What is the locus of a point in space equidistant from the extremities of a given line? A plane.

What is the locus of a line moving so that each point in it traces a right line? In general, a plane; if it move in the direction of its length, a straight line.

What is the locus of a right line parallel to and equidistant from a given line?

What is the locus of a right line intersecting a given line at a constant angle? * A conical surface of revolution.

What is the locus of a semicircle revolving on its diameter?

PROPOSITIONS AND PROBLEMS IN DETERMINING LOCI.

[Note.—The student should be required to give every demonstration *in form*, and in detail. Frequent exercise in writing out demonstrations, is almost the only method of securing a good, independent style in demonstration.]

858. Theo.—*The locus of a point in a plane, equidistant from the extremities of a given line, is a perpendicular to that line at its middle point.*

Sug.—To prove this we have simply to show that every point in such a perpendicular is equidistant from the extremities of the given line, and that no other point has this property (Part II., ***129***).

* That is, an angle which remains of the same size.

859. Prob.—*Find the locus of a point at any constant distance* m *from a straight line.* Of what proposition in PART II. is this the converse?

SUG'S.—To prove the proposition which the answer to this question asserts, it will be necessary to show that every point in the affirmed locus is at the same distance from the given line and that *no other* point is at that distance. We affirm that *the locus is two right lines parallel to the given line and at a distance* m *therefrom.* The formal demonstration is as follows: Let AB be the given line, and OE, OE′, perpendiculars thereto, each equal to m. Through E and E′ draw CD and C′D′ parallel to AB; then is CD, C′D′, the locus required.* For, by Part II. (***156***), *every* point in CD, C′D′, is at the distance m from AB; and we may readily show that any other point, as P or P′, is at a distance greater or less than m from AB. Hence CD, C′D′, is the locus required.

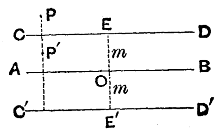

FIG. 445.

860. Theo.—*In a circle, the locus of the centre of a chord parallel to a given line is a diameter.*

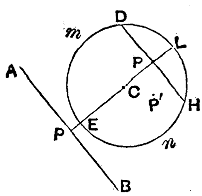

FIG. 446.

DEM.—Let mn be any circle, and AB a given line. Then is the locus of the centre of a chord parallel to AB, a diameter of the circle.

For, let DH be *any* chord parallel to AB. Through the centre of the circle C, and P, the middle point of DH, draw EL. Now EL is *perpendicular* to DH (?), and consequently to AB (?). Then will EL be perpendicular to *any* and every chord parallel to DH (?), and hence will bisect such chord (?). Therefore the locus of the centre of a chord parallel to AB is a diameter.

Again, any point in the circle and out of the line EL is not the middle point of chord parallel to AB. Thus, letting P′ be such a point, draw a chord through P′ parallel to AB. As there can be but one such chord (?), and as EL bisects it (?), P′ is without the diameter (?).

861. Theo.—*The locus of the centre of a circumference passing through two given points is a straight line.*

SUG.—Consult PART II. (***159, 163, 197***). The student should put the argument in form.

862. Theo.—*The locus of the centre of a circle which is tan-*

* It is important that the student think of these two lines as *one* locus, or as *parts* of *one and the same* locus, if this will aid the conception. A locus may consist of any number of detached parts: all that is necessary being that the given conditions be fulfilled. In this respect the word *locus* has a more enlarged meaning than the term *geometrical figure.*

gent to a given circle at a given point, is a straight line passing through the centre of the given circle.

DEM.—Let C be the centre of the given circle, and B the point in the circumference to which the circle * shall be tangent, the locus of whose centre is required. Through B draw TL tangent to the given circle. Now, a circle passing through B, and tangent to the given circle, will have TL for its tangent (?), and as a radius is perpendicular to a tangent at its extremity, and only one perpendicular can be drawn to TL through B, the centre of a circle tangent to the given circle at B must be in this straight line. Moreover, as the given circle is tangent to the right line TL at B, its centre is in the perpendicular AX. Hence AX is the locus required.

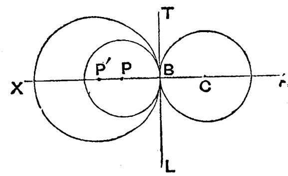

FIG. 447.

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863. Theo.—*The locus of the centre of a circle of given radius* R, *and tangent to a given straight line, is two parallels to this line at a distance* R *therefrom, on each side.* Give proof in form.

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864. Prob.—*Find the locus of the centre of a circle of given radius* R, *whose circumference passes through a given point.* Give proof in form.

---

865. Theo.—*The locus of the centre of a line of constant length, having its extremities in two fixed lines which cut each other at right angles, is the circumference of a circle.*

SUG'S.—Let MN be the length of the given line, and CD, and AB, the two lines intersecting at right angles, in which the extremities of MN are to remain. Now, in whatever position MN may be placed, its middle point, P, is at the same distance ($\frac{1}{2}$MN) from O (?). To show that any point not in this circumference, as p, is not the middle point of a line equal to MN passing through it, and limited by the fixed lines, from p as a centre, with a radius $\frac{1}{2}$MN cut CD, as in C; and from C as a centre with the same radius strike the arc pP. If p is without the circle, CB > MN, if within, less. Hence, the required locus is a circumference whose centre is O, and whose radius is $\frac{1}{2}$MN.

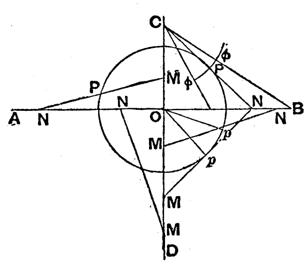

FIG. 448.

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* Observe the form of expression. We say "the circle," and not "the circles," using the term in a *generic sense*, as including all which have the required property, *i. e.*, all which are tangent to the given circle at B.

866. Prob.—*Find the locus of the centre of a chord of constant length, in a given circle.*

Sug.—We say, once again, *always give the proof in form.*

867. Prob.—*Find the locus of the vertex of the right angle of a right angled triangle of a constant hypotenuse.*

868. Prob.—*Find the locus of the middle point of the chord intercepted on a line through a given point, by a given circumference, when the given point is without the circumference, when it is in, and when it is within the circumference.*

869. Prob.—*Find the locus of a point the sum of whose distances from two fixed intersecting lines is constant,* i. e., *is equal to a given line.*

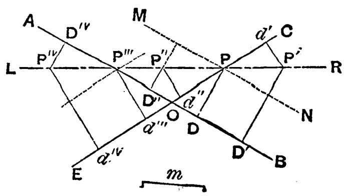

Fig. 449.

Solution.—Let AB and CE be the fixed lines, and m the constant distance. Draw MN parallel to AB, and at a distance m from it. Bisect the angle CPN. Then is LR (a part of) the locus required. For [the student will here show that the sum of the distances from any point in MN to AB and CE, as PD, P′D′ + P′d', P″D″ + P″d'', P‴D‴, PᴵᵛDᴵᵛ − Pᴵᵛd^{iv}, is constant and equal to m], observe that when one of the perpendiculars measuring the distance from a point in the locus, changes from one side to the other of the line on which it is let fall, its *sign* changes. Thus P′D′, PD, P″D″ being considered +, P′d' and PᴵᵛDᴵᵛ are to be considered −. This is a general principle in mathematics. See Part II. (***215***), and foot note.

Finally, LR is only a part of the locus, since there is another line on the opposite side of AB, obtained by drawing the auxiliary MN on that side, which fulfills the same condition. The student should show what the result is when we draw the auxiliary parallel to CE, and on either side of it, also that any point not in one of these lines cannot fulfill the required condition. The complete locus is four indefinite right lines intersecting each other at right angles, so as to inclose a rectangle.

870. Prob.—*Find the locus of a point such that the sum of the squares of its distances from two fixed points shall be equivalent to the square of the distance between the fixed points.*

871. Prob.—*Find the locus of the intersection of two secants*

drawn through the extremities of a fixed diameter in a given circle, one of the secants being always perpendicular to a tangent to the circle at the point where the other cuts it.

SUG'S. P being the point, show that PB = AB, for any position of AP and BP. Hence, any point in the circumference having B for its centre, and AB for its radius, fulfills the conditions. Show that any point out of the circumference does not fulfill the conditions.

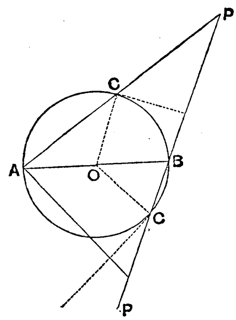

FIG. 450.

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872. Prob.—*Find the locus of the intersection of two lines drawn from the acute angles of a right angled triangle, through the points where the perpendicular to the hypotenuse cuts the opposite sides, or sides produced.*

SUG'S.—The locus of P is required. Prove that APC is always a right angled triangle, wherever the perpendicular EF to the hypotenuse AC is drawn.

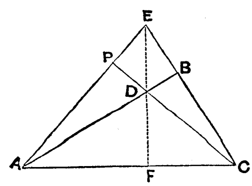

FIG. 451.

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873. Prob.—*Find the locus of a point which divides a line drawn from a fixed point to a fixed line in a fixed ratio.*

SUG'S.—Most problems in finding loci such as are treated in Elementary Plane Geometry, viz., right lines and circles, are readily solved by constructing a few points according to the given conditions, whence we can determine by inspection whether the required locus is a right line or the circumference of a circle; and, having discovered this fact by inspection, it will remain to show *why it should be so.* Thus, in the present problem, O being the fixed point, and AB the fixed line, drawing a few lines, OC, according to the requirements, and dividing them in the same ratio (in the figure 3 : 2), we find a few points P in the locus. We then discover at once that the locus is a right line parallel to AB, and can easily see *why* it should be so.

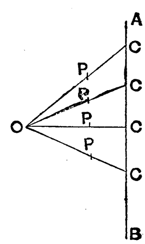

FIG. 452.

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874. Prob.—*Two fixed circumferences intersect: to find the locus of the middle point of the line drawn through one of the points of intersection and terminated by its other intersection with the circumferences.*

SUG'S.—We will first give an example of the course which the mind of the student might take in his efforts to discover the solution. He would naturally draw two *unequal* * circles, as M and N, and, through one of the points of intersection, as A, draw BC, and bisect it at P. It is the locus of P that is desired. Now, suppose the line BC to revolve about A, B passing towards B′, and C towards A. It is the path of the middle point that he seeks. When C reaches A, the line becomes tangent to N, and P is the middle point of the chord AB′. In a similar manner, he sees that the middle point of the chord AC′, tangent to M at A, is also a point in the locus.

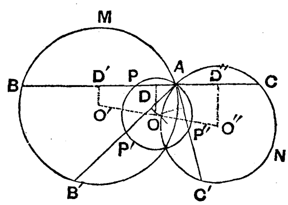

FIG. 453.

Again, he observes that as B moves towards M, and C towards N, P moves towards A, and when AC = AB, P is at A. It now appears probable that the locus of P is a circumference. Proceeding on this hypothesis, he reasons, that, if this is true, AP′ and AP″ are chords of the locus, and, bisecting them with perpendiculars, he will have the centre of the locus. Locating O thus, he observes that it *appears* to be in the line joining the centres O′ and O″, and about midway between them. This leads him to see, whether, by assuming the middle point of O′O″ as the centre of a circle, and OA as a radius, he can prove that any such line as BC drawn through A is bisected by this circumference, as at P. This he can readily prove by means of the perpendiculars OD, O′D′, and O″D″, which bisect the chords AP, AB, and AC. For, since these perpendiculars are parallel, and O′O = OO″, D′D = DD″; whence D′P = AD″, and, adding AP to each, D′A = PD″, or D′B = PD″. Adding to BD′, D′P, and D″C (= AD″ = D′P), there results BP = PC, and the hypothesis is true.

But, *in giving this problem as a recitation*, the student will proceed as follows: Letting M and N be the two fixed circumferences, intersecting at A, join their centres O′ and O″, and bisect O′O″ as at O. With O as a centre and OA as a radius, describe a circle. Then is this circumference the locus required. For, let BC be any secant line passing through A, we may show that P is the middle point of BC. [Having done this, as above, and shown that any point not in this circumference is not the middle of the secant line passing through A, his solution is complete.]

---

875. Prob.—*If the line* AB *is divided at* C, *find the locus of* P, *so that angle* APC = *angle* BPC.

SUG'S.—*In seeking for the solution*, the following would be a natural process. Drawing *any* line, as AB, in the lower part of the figure, taking C, any point in it, and *conceiving* BP and AP drawn so as to make *equal angles* with PC, we would naturally discover that, if a circle were circumscribed about BPA, PC produced would bisect the arc below AB. Thus we discover a ready method of locating P; *i. e.*, in the main figure, bisect AB by a perpendicular, as ED, and

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* *Equal* circles would probably have *special* relations.

with any point on ED as a centre, pass a circumference through A and B. Through D and C draw a line, and P is a point in the locus (?). Any number of points can be found in this way; and, having found a few, as P, P′, P″, P‴, etc., the situation of these will suggest that, *probably*, the locus is the circumference of a circle whose centre is in AB produced. *If this should be the fact,* CP′ is a chord of that circle, and, erecting a perpendicular at the middle point of CP′, its intersection with AB produced, as O, will be the centre of the locus (?). We will now endeavor to prove that *any* point in this circumference, as P′, is so situated that BP′C = CP′A, and that no point out of this circumference has this property. We can readily show that the angle OP′B = OCP′ − BP′C = OCP′ − CP′A (?). But P′AC = OCP′ − CP′A (?). ∴ Triangle OP′B is similar to OP′A (?), and OA : OP′ : : OP′ : OB, or OA : OC : : OC : OB. Now, for *any* point in this circumference, as P, we shall have OA : OP : : OP : OC, since OP = OC, and OA, OC, and OB are constant. Hence, wherever P is taken (in this circumference), the triangle OPB is similar to OPA, angle OPB = PAB, and BPC = CPA. Finally, that no point out of this circumference possesses this property is evident, since the distance of such a point from O would not equal OC, and the angle OP_1B (P_1 being such a point) would not equal P_1AB.

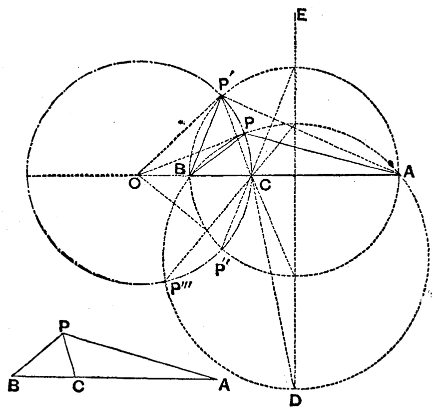

FIG. 454.

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876. *Prob.*—*In a fixed circle, any two chords intersect at right angles in a fixed point; find the locus of the centre of the chord joining their extremities.* Give the proof.

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877. *Prob.*—*Find the locus of the point in space equidistant from three given points.* Give the proof.

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878. *Prob.*—*Find the locus of the point in space equidistant from two given points.* Give the proof.

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879. *Prob.*—*Find the locus of the point in a plane such that the difference of the squares of the distances from it to two fixed points without the plane shall be constant.*

SUG'S.—Conceive the two points without the plane joined by a right line, and a perpendicular to this line drawn from either extremity of it; the point where this perpendicular pierces the plane is a point in the locus (?). The required locus is two parallel straight lines.

880. Prob.—*Find the locus of the middle point of a straight line of constant length, whose extremities remain in two lines at right angles to each other, but which are not in the same plane.* Give the proof.

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881. Prob.—*Find the locus of the point equidistant from two fixed planes.* Give proof.

Sug's.—Consider, 1st, When the fixed planes are parallel; and 2d, when they intersect.

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882. Prob.—*What locus is the intersection of a plane and the surface of a sphere?* Give proof.

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883. Prob.—*What locus is the intersection of the surfaces of two given spheres?*

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884. Prob.—*Find the locus of the point in space such that the ratio of its distance from a given right line to its distance from a fixed point in that line is constant.*

SECTION II.

OF SYMMETRY.

885. Def.—Two points are said to be *symmetrical with respect to a third,* when the right line joining the two points is bisected by the point of reference, called the *Centre of Symmetry.*

886. Def.—Two loci, or two parts of the same locus, are *symmetrical with respect to a point,* when every point in one has its symmetrical point in the other.

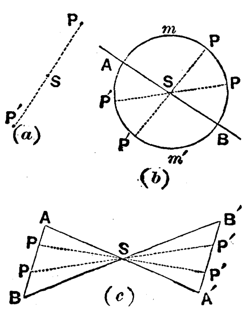

Fig. 455.

Ill's.—In (*a*) P′ is symmetrical with P in respect to S, if S is the middle point of PP′. In (*b*) we observe that the semi-circumference A*m*′B is symmetrical with the semi-circumference A*m*B, in respect to the centre S; for any point P in the latter has a symmetrical point P′ in the former. In (*c*) the triangle A′SB′ is symmetrical with ASB in respect to S (?).

887. Theo.—*The symmetrical of a straight line, with respect to a point, is an equal straight line.*

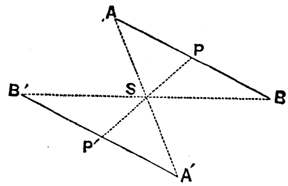

FIG. 456.

SUG'S.—We commence by assuming A′S = AS, and B′S = BS, and drawing B′A′. We then have to show that *any* point in AB, as P, has its symmetrical point P′ in B′A′.

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888. Theo.—*The symmetrical of an angle, with respect to a point, is an equal angle.*

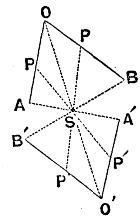

FIG. 457.

SUG'S.—To show the symmetrical of AOB, with respect to S, take A′S = AS, B′S = BS, O′S = OS, and draw O′B′, O′A′. Then show that any point in OA has its symmetrical in O′A′, and any point in OB has its symmetrical in O′B′. Hence, A′O′B′ is the symmetrical of AOB, with respect to S.

Then apply AOB to A′O′B′ and show that these symmetricals are equal.

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889. Prob.—*Having given a polygon, to draw its symmetrical with respect to a given point.*

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890. Theo.—*Any polygon is equal to its symmetrical with respect to a given point.*

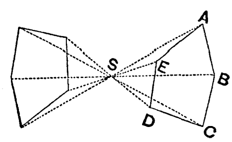


FIG. 458.

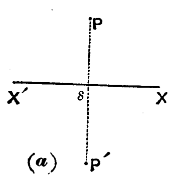
SUG'S.—Proof by revolving the figure about S, keeping it in its own plane, each line, as AS, ES, etc., passing through 180°.

891. DEF.—The lines AS, ES, BS, etc., are *radii of symmetry.*

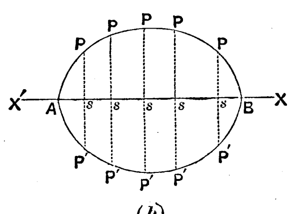
892. DEF.—Two points are *symmetrical with respect to a straight line* in the same plane, when the straight line which joins them is bisected at right angles by the line of reference, called the *Axis of Symmetry.*

893. DEF.—Two loci, or two parts of the same locus, are *symmetrical with respect to a straight line* when every point in the one has its symmetrical point in the other.

ILL'S.—Thus in (*a*) P and P′ are symmetrical points with respect to the right line X′X, P′*s* = P*s* and X′X is perpendicular to PP′. So the part of (*b*) above X′X is symmetrical with the part below, *i. e.*, the curve is symmetrical with respect to X′X.



(*a*)



(*b*)

FIG. 459.

894. Theo.—*The symmetrical of a straight line, with respect to a rectilinear axis of symmetry, is an equal right line.*

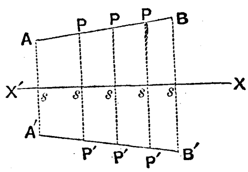


FIG. 460.

DEM.—Let AB be any straight line, and X′X the axis. Let fall the perpendiculars A*s* and B*s*, and produce them till A′*s* = A*s*, and B′*s* = B*s*. Then is AB = A′B′, the symmetrical of AB. For, taking P, *any* point in AB, letting fall P*s*, and producing it to P′, the point P′ is symmetrical with P; since revolving *s*A′B′*s* upon X′X, A′B′ will coincide with AB, and P′ will fall at P. Hence A′B′ = AB, and every point in AB has its symmetrical point in A′B′ (*893*).

895. COR. 1.—*If two straight lines intersect, their symmetricals intersect, and the points of intersection are symmetrical.*

The student should show how this follows from the proposition.

896. COR. 2.—*Two rectilinear symmetricals meet the axis in the same point, and make equal angles therewith.*

Student give proof.

897. DEF.—A *Trapezoid* like ABB′A′, having its non-parallel sides equal, is called *Isosceles.*

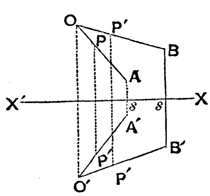


FIG. 461.

898. Theo.—*The symmetrical of an angle, with respect to a rectilinear axis of symmetry, is an equal angle.*

SUG'S.—The student should be able to give the demonstration from the figure, in a manner altogether similar to the preceding; or, drawing AB and A′B′, he can base it upon the preceding.

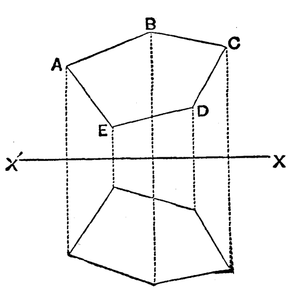


FIG. 462.

899. Prob.—*Having given a polygon, to draw its symmetrical with respect to a given axis.*

900. Theo.—*Any plane figure is equal to its symmetrical, with reference to a rectilinear axis.*

Proof by applying one to the other by revolution.

901. Def.—Two loci in space, or two parts of the same locus (planes or solids), are symmetrical with respect to a point, when every point in one has a corresponding point in the other, such that the line joining them is bisected by the point called the centre of symmetry.

Ill's.—Symmetrical triedrals (Part II., ***432***) afford an illustration of solids symmetrical with respect to a point—the vertex. The two hemispheres into which a great circle divides a sphere are symmetrical parts of the solid (sphere) with reference to the centre.

902. Def.—Two points in space are symmetrical with respect to a plane called the *Plane of Symmetry*, when the line joining the points is perpendicular to the plane and bisected by it.

903. Def.—Two loci in space (planes or solids), or two parts of the same locus, are symmetrical with respect to a plane when every point in one has its symmetrical point in the other.

904. Def.—The corresponding (symmetrical) parts of symmetrical figures are called *Homologous* parts.

---

905. Theo.—*The symmetrical of a right line, with respect to a plane, is an equal right line.*

Dem.—Let AB be any right line, and MN the plane of symmetry. Let fall the perpendiculars Bb, Aa, upon the plane, produce them, making $B'b$ = Bb, $A'a$ = Aa, and join A′ and B′. Then A′B′ = AB, and is its symmetrical. For ABB′A′ being a plane figure (?) and ab, the intersection of this plane with MN, being a right line bisecting AA′ and BB′ at right angles (?), we may revolve abB′A′ upon ab and bring A′B′ into coincidence with AB. Hence A′B′ = AB. Again, P being any point in AB, draw PP′ perpendicular to ab, and upon revolution P′ will fall in P, and P′s = Ps. Hence, every point in AB has its symmetrical in A′B′, and the latter line is the symmetrical of the former.

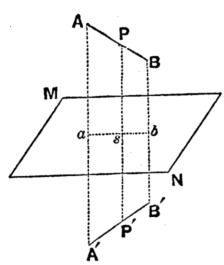


Fig. 463.

906. Cor. 1.—*A right line and its symmetrical, with respect to a plane, pierce the plane at the same point.*

Student give proof.

---

907. Theo.—*The symmetrical of a plane angle, with respect to a plane, is an equal plane angle.*

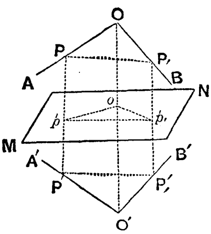


Fig. 464.

Dem.—Let AOB be any plane angle, and MN the plane of symmetry. Let P be any point in AO, and P′ any point in OB. Let O′ be the symmetrical of O, P′ of P, and P_1' of P_1; then is A′O′ the symmetrical of AO, O′B′ of OB, and angle A′O′B′ of AOB. Now by the preceding proposition the two triangles POP, and P′O′P′, are mutually equilateral, whence AOB = its symmetrical A′O′B′.

Query.—When will the triangle *pop′* exist, and when not?

---

908. **Theo.**—*Any plane polygon has for its symmetrical, with reference to a plane, an equal plane polygon.*

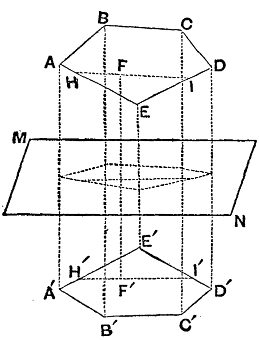


Fig. 465.

Sug's.—ABCDE being any plane polygon, and MN the plane of symmetry, by constructing A′,B′,C′,D′,E′ symmetrical with A, B, C, D, E, we have by the preceding propositions A′B′C′D′E′ equilateral and equiangular with ABCDE; whence it only remains to show that A′B′C′D′E′ is a plane (not a warped) surface. Let F be any point in the angle AED, draw HI, and let H′ and I′ be the symmetricals of H and I (*895*). Draw H′I′. Then is the symmetrical of F in H′I′ (?), as at F′. Now, every point in HF within the angle BAE has its symmetrical in H′F′ (*894*). Thus, by taking three points, not in a straight line, in the angle BAE, we can show that their symmetricals are in the plane B′A′E′, and also in A′E′D′. In like manner, all the angles of A′B′C′D′E′ can be shown to be in the same plane.

909. Cor.—*If two planes intersect, their symmetricals intersect, and the two intersections are symmetrical right lines.*

The student should show how this grows out of the proposition.

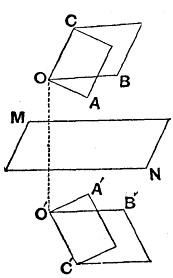


Fig. 466.

910. **Theo.**—*The symmetrical of a diedral is an equal diedral.*

Dem. AOB being the measure of the diedral A-OC-B, and A′-O′C′-B′ the symmetrical diedral, and O′ the symmetrical of O, the symmetrical of AO being A′O′, the angle A′O′C′ is right, and in like manner B′O′ being the symmetrical of BO, B′O′C′ is right. But BOA = B′O′A′ (?), whence the diedrals are equal.

911. Theo.*—Two polyedrons, symmetrical with respect to a plane, have their faces equal, each to each, and their homologous solid angles symmetrical.*

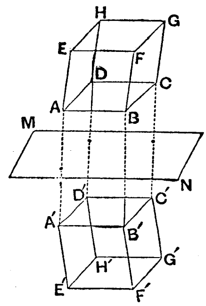


Fig. 467.

Sug's.—This is an immediate consequence of preceding propositions. Thus E′ being the symmetrical solid homologous with E, the homologous plane faces including them are equal (*908*). Again, the facial angles being equal, but not similarly disposed, the solid angles are symmetrical.

912. Cor.—*Two symmetrical polyedrons can be decomposed into the same number of tetraedrons, symmetrical each to each.*

For we can decompose one of the polyedrons into tetraedrons having for their common vertex one of the vertices of this polyedron, and each of these tetraedrons will have its symmetrical in the other.

---

913. Theo.*—Two symmetrical polyedrons are equivalent.*

Dem.—From the last corollary it will appear that it is sufficient for the demonstration of this proposition to show that two symmetrical tetraedrons are equivalent (?). Let S-ABC, and S′-ABC be two tetraedrons symmetrical with respect to their common base. They have a common base and equal altitudes (?), hence they are equivalent.

---

914. General Scholium.—We may speak of two loci, or two parts of the same locus, as symmetrical with respect to a line or plane, whenever all the points in one have symmetrical points in the other, even though the line joining the symmetrical points be not *perpendicular* to the axis, or the plane, of symmetry; observing, however, that this line is always bisected by the axis or plane. Thus, the ellipse in the figure is symmetrically divided by the line X′X, since every point in one portion has a symmetrical point in the other, as $Ps = P's$, for every point in the curve. In such a case the parts cannot be brought into coincidence by simple revolution: one part must be reversed.

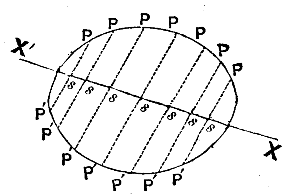


Fig. 468.

SECTION III.

OF MAXIMA AND MINIMA.

915. Def.—***A Maximum*** value of a magnitude conceived to vary continuously in some specified way, is a value which is greater than the preceding and succeeding values of the magnitude.

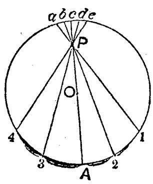


Fig. 469.

Ill's.—Thus, suppose in a given circle, a chord passing through a fixed point, P, revolves so as to take successively the positions $1a$, $2b$, Ac, $3d$, $4e$, etc. It is at a maximum when it passes through the centre, as Ac. The chord is the magnitude which is conceived to vary in the way specified, and Ac is a value greater than the preceding and the succeeding values. Again, conceive a circle to be compressed or extended, as in the direction mn, so as to take the forms indicated by the dotted lines, its *area* will be diminished, the perimeter remaining the same. That is, of all figures of a given perimeter, the circle has the maximum area.

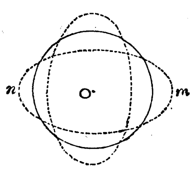


Fig. 470.

916. Def.—***A Minimum*** value of a magnitude conceived to vary continuously in some specified way, is a value which is less than the preceding and succeeding values of the magnitude.

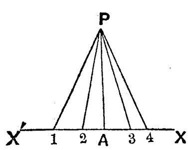


Fig. 471.

Ill's.—Thus, conceive the varying magnitude to be a straight line from the fixed point P to the fixed line X'X; that is, suppose such a line to start from some position P1, and move through the successive positions P2, PA, P3, P4. PA is a minimum, since it is less than the preceding and succeeding values.

PROPOSITIONS CONCERNING MAXIMA AND MINIMA.

917. ***Axiom.***—*The minimum distance between two points is a straight line.*

---

918. ***Theo.***—*The minimum distance from a point to a line is a straight line perpendicular to the given line.*

Student give proof.

---

919. ***Theo.***—*The maximum line which can be inscribed in a given circle is a diameter.*

Proof based on the fact that the hypotenuse of a right angled triangle is the greatest side.

---

920. Theo.—*The sum of the distances from two points on the same side of a line, to a point in the line, all being in the same plane, is a minimum when the lines measuring the distances make equal angles with the given line.*

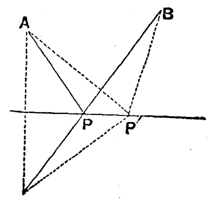


FIG. 472.

Student prove $AP + BP < AP' + BP'$.

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921. Theo.—*If a triangle have a constant base and altitude, its vertical angle is a maximum when the triangle is isosceles.*

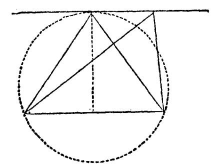
FIG. 473.

SUG.—By what is the vertical angle measured?

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922. Theo.—*The base and area of a triangle being constant, its perimeter is a minimum when the triangle is isosceles.*

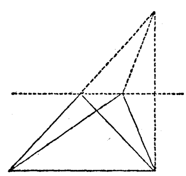
FIG. 474.

SUG'S.—The area and base being constant, the vertex remains in a line parallel to the base, for all values of the other sides. The figure will suggest the demonstration, which is based on the fact that any side of a triangle is less than the sum of the other two.

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923. Theo.—*The difference between the distances from two points on opposite sides of a fixed line to a point in that line, is a maximum, when the lines measuring these distances make equal angles with the fixed line.*

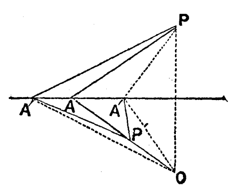

FIG. 475.

SUG'S. $P'O = AP - AP'$; but $P'O > A'O\ (= A'P) - A'P'$.

QUERY.—Having the points P, P', and the fixed line given, how is the point A found by geometrical construction?

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924. Theo.—*The lengths of two sides of a triangle being constant, the area is a maximum when the included angle is right.*

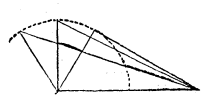
FIG. 476.

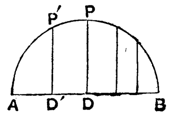

FIG. 477.

925. Theo.—*The sum of two adjacent sides of a rectangle being constant* (AB), *the area is a maximum when the sides are equal.*

ISOPERIMETRY.

926. *Isoperimetric* Figures are such as have equal perimeters, *i. e.*, bounding lines of equal length.

Problems in isoperimetry are a species of problems in Maxima and Minima. Thus, of all figures whose perimeters are m (say 10 inches), to find that which has the greatest area, is a problem in isoperimetry. Again, what must be the form of a pentagon whose perimeter is m, in order that its area may be a maximum?

927. Theo.—*Of isoperimetric triangles with a constant base, the isosceles is a maximum.*

SUG'S.—By means of the figure to Theorem (**921**), we can readily show that any triangle having the same base as the isosceles triangle, and its vertex either in or beyond the line through the vertex of the isosceles triangle and parallel to its base, has a greater perimeter than the isosceles triangle. Hence, the isoperimetric triangle on the given base has its vertex below this parallel, except when isosceles; and consequently the isosceles is the maximum.

928. COR.—*Of isoperimetric triangles, the equilateral has the maximum area* (?)

929. Prob.—*Given any triangle with a constant base, to construct the maximum isoperimetric triangle.*

930. Prob.—*Given any triangle, to construct the maximum isoperimetric triangle.*

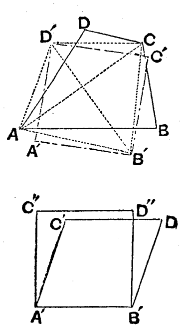

FIG. 478.

931. Theo.—*Of isoperimetric quadrilaterals, the square has the maximum area.*

DEM.—Let ABCD be any quadrilateral. If AD is not equal to DC, the triangle ADC can be replaced by the isoperimetric triangle AD′C, and the area of the quadrilateral increased. So ABC can be replaced by AB′C. Therefore AB′C′D > ABCD. In like manner if AD′ is not equal to AB′, D′AB′ can be replaced by the maximum isoperimetric triangle D′A′B′. So also D′CB′ can be replaced by D′C′B′. Therefore A′B′C′D′ > AB′CD′ > ABCD. Now, A′B′C′D′ is a rhombus (?), and the student can show that the square on A′B′ is greater than *any* rhombus with the same side.

932. Prob.—*Having given a quadrilateral, to construct the maximum isoperimetric quadrilateral.*

---

933. Theo.—*Of isoperimetric quadrilaterals with a constant base, the maximum has its three remaining sides equal each to each, and the angles which they include equal.*

Dem.—Let ABCD be the maximum isoperimetric quadrilateral on the base AD, then AB = BC = CD, and angle ABC = BCD. For, if AB is not equal to BC, draw AC, and replacing the triangle ABC with its isoperimetric isosceles triangle, we shall have a quadrilateral isoperimetric with ABCD, and greater than ABCD, *i. e.*, greater than the maximum, which is absurd.

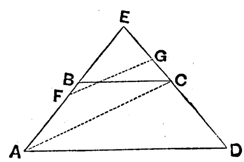

Fig. 479.

Again, if angle ABC is not equal to BCD, let ABC < BCD, whence BCE < EBC, and BE < EC. Take EF = EC, and EG = EB, whence the triangles FEG and BEC are equal, and FG = BC. Also, since AB + BC + CD = AE + ED − (EB + EC) + BC, and AF + FG + CD = AE + ED − (FE + EG) + FG, it follows that AFGD and ABCD are isoperimetrical, and, since ABCD = AED − BEC, and AFGD = AED − FEG, that AFGD and ABCD are equal. Therefore, AFGD is a maximum, and by the preceding part of the demonstration AF = FG = BC = AB, which is absurd; and there can be no inequality between angles ABC and BCD.

---

934. Theo.—*Of isoperimetric polygons of a given number of sides, the regular polygon has the maximum area.*

Dem.—First, the polygon must be equilateral; for, if any two adjacent sides, as AB, BC, are unequal, the triangle ABC can be replaced by its isoperimetric isosceles triangle, and thus the area of the polygon be increased.

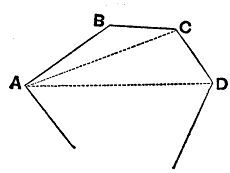

Fig. 480.

Second, the polygon must be equiangular; for, if any two adjacent angles, as B and C, are unequal, the quadrilateral ABCD can be replaced by its isoperimetric quadrilateral with B = C, and thus the area of the polygon be increased.

---

935. Theo.—*Of isoperimetric regular polygons, the one of the greater number of sides is the greater.*

Dem.—Let ABC be an equilateral (regular) triangle. Join any vertex, as A, with any point, as D, in the opposite side. Replace the triangle ACD with the isosceles triangle AED. Then is the quadrilateral ABDE > the triangle ABC.

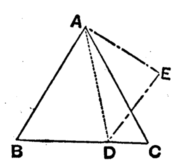

Fig. 481.

But, of isoperimetric quadrilaterals, the regular (the square) is the greater. Hence, the regular quadrilateral (the square) isoperimetric with the triangle ABC, is greater than

the triangle. In the same manner the regular pentagon isoperimetric with the square can be shown greater than the square; and thus on, *ad libitum.*

936. Cor.—*Of plane isoperimetric figures, the circle has the maximum area,* since it is the limiting form of the regular polygon, as the number of its sides is indefinitely increased.

SECTION IV.

OF TRANSVERSALS.

937. Def.—***A Transversal*** is a line cutting a system of lines. A transversal of a triangle is a line cutting its sides;* it either cuts two sides and the third side produced, or the three sides produced. In speaking of the transversal of a triangle (or polygon), the distances on any side (or side produced) from the intersection of the transversal with that side to the angles, are *Segments.* Of these there are six. *Adjacent segments* are such as have an extremity of each at the same point. *Non-adjacent segments* are such as have no extremity common.

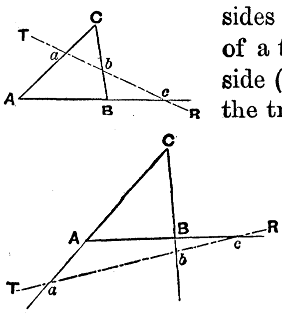

Fig. 482.

Ill's. TR is a transversal of the triangle ABC; aA, aC, bC, bB, cA, cB are adjacent segments two and two; aC, bB, cA, and aA, bC, cB are the two groups of non-adjacent segments.

938. The two Fundamental Propositions of the Theory of Transversals.

939. ***Theo.***—*The product of three non-adjacent segments of the sides of a triangle cut by a transversal, is equal to the product of the other three.*

Dem.—ABC being cut by the transversal TR, $a\mathrm{A} \times b\mathrm{C} \times c\mathrm{B} = a\mathrm{C} \times b\mathrm{B} \times c\mathrm{A}$. Draw BD parallel to AC, and from the similar triangles we have

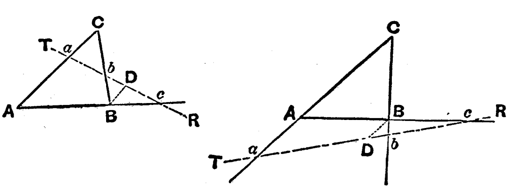

Fig. 483.

$$\frac{a\mathrm{A}}{\mathrm{BD}} = \frac{c\mathrm{A}}{c\mathrm{B}}, \text{ and } \frac{\mathrm{DB}}{b\mathrm{B}} = \frac{a\mathrm{C}}{b\mathrm{C}};$$

whence, multiplying,

$$\frac{a\mathrm{A}}{b\mathrm{B}} = \frac{a\mathrm{C} \times c\mathrm{A}}{b\mathrm{C} \times c\mathrm{B}},$$

or $a\mathrm{A} \times b\mathrm{C} \times c\mathrm{B} = a\mathrm{C} \times b\mathrm{B} \times c\mathrm{A}$.

* Or, sides produced—this expression being usually omitted in higher Geometry as all lines are to be considered indefinite unless limited in the problem.

940. COR.—Conversely, *If three points be taken in the sides of a triangle* (*as* a, b, c) *such that the product of three non-adjacent segments equals the product of the other three, the points are in the same straight line.*

For, passing a line through a and b, let it cut the third side in c'. Then, by the proposition, $a\mathsf{A} \times b\mathsf{C} \times c'\mathsf{B} = a\mathsf{C} \times b\mathsf{B} \times c'\mathsf{A}$. But, by hypothesis $a\mathsf{A} \times b\mathsf{C} \times c\mathsf{B} = a\mathsf{C} \times b\mathsf{B} \times c\mathsf{A}$. Whence $\frac{c\mathsf{B}}{c'\mathsf{B}} = \frac{c\mathsf{A}}{c'\mathsf{A}}$, and c and c' must coincide.

SCH.—This theorem is known among mathematicians as *The Ptolemaic Theorem*, and is usually attributed to Claudius Ptolemy, an Egyptian mathematician and philosopher who flourished in Alexandria during the first half of the second century. But it is thought to be more properly due to *Menelaus*, who lived a century before Ptolemy.

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941. Theo.—*The three angle-transversals* of a triangle, passing through a common point, divide the sides into segments such that the product of three non-adjacent segments equals the product of the other three.*

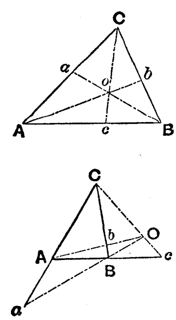

FIG. 484.

DEM.—From the triangle $\mathsf{AC}c$ cut by the transversal $a\mathsf{B}$, we have $a\mathsf{A} \times \mathsf{CO} \times c\mathsf{B} = \mathsf{AB} \times a\mathsf{C} \times \mathsf{O}c$; and from $\mathsf{CB}c$ cut by $b\mathsf{A}$, $\mathsf{O}c \times b\mathsf{C} \times \mathsf{AB} = \mathsf{CO} \times b\mathsf{B} \times c\mathsf{A}$. Multiplying, we obtain $a\mathsf{A} \times b\mathsf{C} \times c\mathsf{B} = a\mathsf{C} \times b\mathsf{B} \times c\mathsf{A}$.

942. COR. 1.—Conversely, *If the three angle-transversals of a triangle divide the sides into segments such that the product of three non-adjacent segments equals the product of the other three, the transversals pass through a common point.*

For, the sides being divided at a, b, and c, so that $a\mathsf{A} \times b\mathsf{C} \times c\mathsf{B} = a\mathsf{C} \times b\mathsf{B} \times c\mathsf{A}$, draw $\mathsf{C}c$, and $\mathsf{A}b$, and let O be their intersection. Now, let a' be the point in which BO cuts AC. Then, by the proposition, $a'\mathsf{A} \times b\mathsf{C} \times c\mathsf{B} = a'\mathsf{C} \times b\mathsf{B} \times c\mathsf{A}$.

$$\text{Whence } \frac{a\mathsf{A}}{a'\mathsf{A}} = \frac{a\mathsf{C}}{a'\mathsf{C}}, \text{ and } a \text{ and } a' \text{ coincide.}$$

943. COR. 2.—*If any one of the sides is bisected, the line joining the other points of division is parallel to this side.*

For, let $b\mathsf{C} = b\mathsf{B}$. Then $a\mathsf{A} \times b\mathsf{C} \times c\mathsf{B} = a\mathsf{C} \times b\mathsf{B} \times c\mathsf{A}$, becomes

$$a\mathsf{A} \times c\mathsf{B} = a\mathsf{C} \times c\mathsf{A}; \text{ or } \mathsf{A}a : a\mathsf{C} :: \mathsf{A}c : c\mathsf{B}.$$

QUERY.—How does this apply to the second figure?

944. COR. 3.—*If the line joining two points of division is parallel to the third side, the latter side is bisected.*

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* The transversals passing through the angles.

For, if ab is parallel to AB, aC : bC : : aA : bB, whence $a\text{C} \times b\text{B} = b\text{C} \times a\text{A}$. And, since $a\text{A} \times b\text{C} \times c\text{B} = a\text{C} \times b\text{B} \times c\text{A}$, $c\text{A} = c\text{B}$.

945. We will now give a few problems to illustrate the use of the theory of transversals.

946. Prob.—*To show that the medial lines of a triangle pass through a common point.*

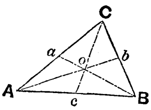

Fig. 485.

Solution.—Since $a\text{A} = a\text{C}$, $b\text{C} = b\text{B}$, and $c\text{B} = c\text{A}$, by multiplying, we have $a\text{A} \times b\text{C} \times c\text{B} = a\text{C} \times b\text{B} \times c\text{A}$; whence by the last corollary these transversals pass through a common point.

947. Prob.—*To show that the bisectors of the angles of a triangle pass through a common point.*

Solution.—In the last figure let aB, bA, cC be the bisectors. Then $\frac{a\text{A}}{a\text{C}} = \frac{\text{AB}}{\text{CB}}$, $\frac{b\text{C}}{b\text{B}} = \frac{\text{AC}}{\text{AB}}$, $\frac{c\text{B}}{c\text{A}} = \frac{\text{CB}}{\text{AC}}$; multiplying, $\frac{a\text{A} \times b\text{C} \times c\text{B}}{a\text{C} \times b\text{B} \times c\text{A}} = 1$, or $a\text{A} \times b\text{C} \times c\text{B} = a\text{C} \times b\text{B} \times c\text{A}$. Therefore these transversals pass through a common point.

948. Prob.—*To show that the altitudes of a triangle pass through a common point.*

Sug's.—In the last figure, if aB, bA, cC, were the perpendiculars, there would be three pairs of similar triangles giving $\frac{a\text{A}}{b\text{B}} = \frac{\text{AO}}{\text{BO}}$, $\frac{b\text{C}}{c\text{A}} = \frac{\text{CO}}{\text{AO}}$, $\frac{c\text{B}}{a\text{C}} = \frac{\text{OB}}{\text{CO}}$; whence, as in the last.

949. Prob.—*To show that the angle-transversals terminating in the points of tangency of the sides of the triangle with its inscribed circle, pass through a common point.*

Sug's.—In the last figure, if a, b, c were the points of tangency we should have $a\text{A} = c\text{A}$, $b\text{C} = a\text{C}$, $c\text{B} = b\text{B}$; whence $a\text{A} \times b\text{C} \times c\text{B} = a\text{C} \times b\text{B} \times c\text{A}$. Which shows that the transversals pass through a common point.

950. Theo.—*If two sides of a triangle are divided proportionally, starting from the vertex, the angle-transversals from the extremities of the other side to the corresponding points of division, intersect in the medial line to this third side.*

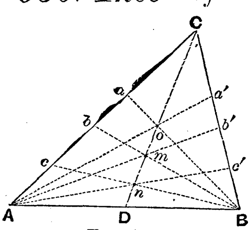

Fig. 486.

Dem.—Since AC and CB are divided proportionally at a and a', $a\text{A} \times a'\text{C} = a\text{C} \times a'\text{B}$; and as $\text{DB} = \text{DA}$, $a\text{A} \times a'\text{C} \times \text{DB} = a\text{C} \times a'\text{B} \times \text{DA}$, the angle-transversals Aa', Ba intersect in CD. The same may be shown of any other angle-transversals from A and B, dividing CB and CA proportionally.

951. Cor.—*In any trapezoid the transversal passing through the intersection of the diagonals, and the intersection of the non-parallel sides, bisects the parallel sides.*

Sug.—Joining aa' in the last figure, CD is such a transversal. The student will readily see the connection with the proposition.

---

952. Prob.—*Through a given point to draw a line which shall meet two given lines at their intersection in an invisible, inaccessible point.*

Solution.—Let Mm, Nn be the two given lines which meet in the invisible, inaccessible point S, and P the given point through which a line is to be located which will meet Mm, Nn in S. Through P draw any convenient transversal, as BF, and any other meeting this, as AF. Now, considering MS as a transversal of the triangle CDF, we have $AF \times BC \times SD = AD \times BF \times SC$; whence $\frac{SD}{SC} = \frac{AD \times BF}{AF \times BC}$. But, HD being drawn parallel to BF, we have $\frac{HD}{PC} = \frac{SD}{SC} = \frac{AD \times BF}{AF \times BC}$, or $HD = \frac{AD \times BF \times PC}{AF \times BC}$; whence HD is known, as AD, BF, PC, AF, BC can be measured. The points P and H determine the required line.

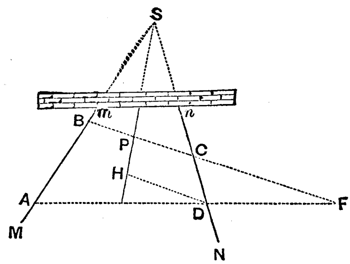

Fig. 487.

953. Def.—The ***Complete Quadrilateral*** is the figure formed by four lines meeting in six points. The complete quadrilateral has three diagonals.

Ill.—ABCDEF is a complete quadrilateral, and its diagonals are CF, BD, and AE, the latter being spoken of as the third or exterior diagonal.

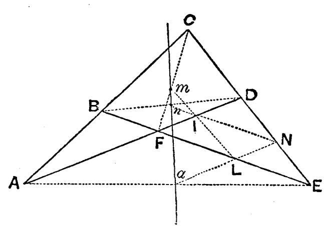

Fig. 488.

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954. Theo.—*The middle points of the three diagonals of a complete quadrilateral are in the same straight line.*

Dem.—m, n, o being the centres of the diagonals of the complete quadrilateral, in the preceding figure, are in the same straight line. Bisect the sides of the triangle FDE, as at I, N, L, and draw IN, IL, LN. Since IN is parallel to BE, and bisects DF and DE, it also bisects DB (?) and hence passes through n. For like reasons IL passes through m, and LN through o. Now, AC being a transversal of the triangle FDE gives $CD \times BE \times AF = CE \times BF \times AD$. Therefore,

noticing that $\frac{1}{2}$CD $= m$I, $\frac{1}{2}$BE $= n$N, $\frac{1}{2}$AF $= o$L, $\frac{1}{2}$CE $= m$L, $\frac{1}{2}$BF $= n$I, and $\frac{1}{2}$AD $= o$N, we have mI $\times$ nN $\times$ oL $= m$L $\times$ nI $\times$ oN. Hence these three points m, n, o lie in a transversal to the triangle ILN.

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955. Theo.—*In the complete quadrilateral, any diagonal is divided harmonically by the other two.*

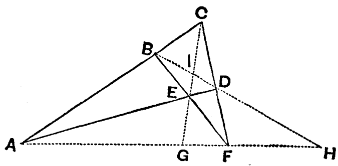

FIG. 489.

DEM.—Thus, AFH is divided harmonically at G and H. For, considering BH as a transversal of the triangle ACF, we have HF $\times$ DC $\times$ BA $=$ HA $\times$ DF $\times$ BC. And CG, AD, FB being angle-transversals of the same triangle, we have GF $\times$ BA $\times$ DC $=$ GA $\times$ BC $\times$ DF. Whence, dividing, $\frac{HF}{GF} = \frac{HA}{GA}$, *i.e.*, AH is divided harmonically. Again, if CH is drawn, CA, CG, CF, CH constitute a harmonic pencil, and BH a, transversal of it, is cut harmonically at B, I, D, H. Finally, if F and I be joined, FH (or FA), FB, FI, FD constitute a harmonically, and hence CG is cut harmonically at C, I, E, G.

956. COR.—An angle-transversal of a triangle, and a line passing through the feet of the other angle-transversals, divide the third side harmonically.

SECTION V.

HARMONIC PROPORTION AND HARMONIC PENCILS.

957. DEF.—Three quantities are in *Harmonic Proportion* when the difference between the first and second is to the difference between the second and third, as the first is to the third.

ILL.—6, 4, 3 are in harmonic proportion, since $6 - 4 : 4 - 3 :: 6 : 3$. In general, a, b, c are in harmonic proportion, if $a - b : b - c :: a : c$.

---

958. Theo.—*If a given line be divided internally and externally in the same geometric ratio, the distance between the points of division is a harmonic mean between the distances of these points from the extremity of the given line, not included between the points.*

DEM. — Let AB be the given line; and let O and O′ be so taken that AO : BO :: AO′ : BO′; then is OO′ a harmonic mean between AO′ and BO′. For AO $=$ AO′ $-$ OO′, and BO $=$ OO′ $-$ BO′; whence AO′, OO′, and BO′ are such that AO $-$ OO′ : OO′ $-$ BO′ :: AO′ : BO′, that is, they are in harmonic proportion.

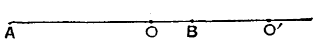

FIG. 490.

959. COR. 1.—AO, AB, *and* AO′ *are in harmonic proportion*, i.e., AB *is a harmonic mean between* AO *and* AO′.

For AB − AO (= BO) : AO′ − AB (= BO′) : : AO : AO′.

960. COR. 2.—*When* AO, OO′, BO′ *are in harmonic proportion,* AO × BO′ = BO × AO′.

961. COR. 3.—Conversely, *When a line is divided into three parts such that the rectangle of the extreme parts equals the rectangle of the mean part into the whole line, the line is divided harmonically.*

Thus, let AO′ be the line, and AO × BO′ = BO × AO′; then AO : BO : : AO′ : BO′, whence, by the proposition, OO′ is a harmonic mean between AO′ and BO′.

DEF.—The points O and O′ are called *Harmonic Conjugates.*

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962. Theo.—*If two lines be drawn, one bisecting the interior and the other the adjacent exterior angle of a triangle, and meeting the opposite side,* they divide this line harmonically.*

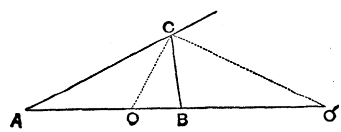

FIG. 491.

SUG.—By means of (*358, 359,* PART II.) the student will be enabled to establish the relation AO : BO : : AO′ : BO′, whence, by the last proposition, AO, OO′, AO′ are in harmonic proportion.

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963. Prob.—*Given a right line to locate two harmonic conjugate points.*

SOLUTION.—Let AB be the line. O may be taken at pleasure between A and B. We are then to find O′, so that AO : BO : : AO′ : BO′. Taking this by division, we have AO − BO : BO : : AO′ − BO′ (= AB) : BO′. The first three terms being known, the other can be constructed. Or, we may first locate O′ at pleasure, and then find O.

---

964. Theo.—*If from the given point* C *in a line the distances* CO, CB, CO′ *be taken in the same direction, so that* CO × CO′ = $\overline{\text{CB}}^2$; *and if* CA = CB *be taken in the opposite direction,* AO′ *will be divided harmonically at* O *and* B.

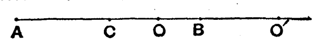

FIG. 492.

DEM. — From CO × CO′ = $\overline{\text{CB}}^2$, we readily write CO : CB : : CB : CO′, CB + CO (= AO) : CB − CO (= BO) : : CO′ + CB (= AO′) : CO − CB (= BO′).

* The bisector of the exterior angle meets the side *produced;* but in higher geometry, as it is always understood that lines are indefinite unless limited by hypothesis, such specifications are deemed unnecessary.

965. Cor. 1.—Conversely, *If a line* AO′ *be cut harmonically at* O *and* B, *and either of the harmonic means be bisected, as* AB *at* C, *the three segments* CO, CB, CO′ *will be in geometric proportion.*

For, since $AO' : BO' :: AO : BO$, $AO' + BO' : AO' - BO' :: AO + BO : AO - BO$, or $2CO' : 2CB :: 2CB : 2CO$, and $CO' : CB :: CB : CO$.

966. Cor. 2.—In a given line, as AB, as O approaches the centre C, O′ recedes, and when O is at C, O′ is at infinity, since $CO' = \frac{\overline{CB}^2}{CO}$.

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967. ***Theo.***—*The geometric mean between two lines is also the geometric mean between their arithmetic and harmonic means.*

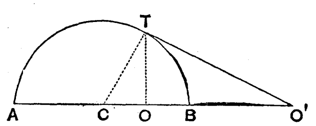

Fig. 493.

Dem.—Let AO′ and BO′ be the two lines. On their difference, AB, draw a semicircle, draw the tangent O′T and let fall the perpendicular TO. Then O and O′ are harmonic conjugates, since $CO \times CO' = \overline{CB}^2$ (?), CO′ is the arithmetic mean (that is, $\frac{1}{2}$ the sum) of AO′ and BO′ (?) and TO′ is the geometric mean (?). $\therefore AO' : TO' :: TO' : BO'$ (?).

Queries.—Which is the greatest, in general, the arithmetic, geometric, or harmonic mean between two quantities? Are they ever equal?

968. Sch.—*This proposition* affords a ready method of finding either of the harmonic conjugates O or O′, when the other is given. The student will show how.

969. Cor. 1.—*The rectangle of the harmonic means and the sum of the extremes, is equivalent to twice the rectangle of the extremes.*

For, $CO' \times OO' = \overline{TO'}^2 = AO' \times BO'$, whence $2CO' \times OO' = 2AO' \times BO'$; and, since $2CO' = AO' + BO'$, $(AO' + BO') \times OO' = 2AO' \times BO'$.

970. Cor. 2.—*The rectangle of the harmonic mean and the difference of the differences of the 1st and 2nd, and the 2nd and 3rd, is equivalent to twice the rectangle of these differences.*

That is, $OO' \times [(AO' - OO') - (OO' - BO')] = 2(OO' - AO')\ (OO' - BO')$, or $OO' \times (AO - BO) = 2AO \times BO$. Let the student give the proof.

971. Cor. 3.—*If three quantities are in harmonic proportion their reciprocals are in arithmetic proportion* (*i.e.*, the difference between the 1st and 2nd equals the difference between the 2nd and 3rd).

For, from AO′, OO′, BO, we have the reciprocals $\frac{1}{AO'}$, $\frac{1}{OO'}$, $\frac{1}{BO'}$. Now $\frac{1}{OO'} - \frac{1}{AO'} = \frac{AO' - OO'}{OO' \times AO'} = \frac{AO}{OO' \times AO'}$; and $\frac{1}{BO'} - \frac{1}{OO'} = \frac{OO' - BO'}{OO' \times BO'}$

$= \frac{BO}{OO' \times BO'}$. But $\frac{AO}{OO' \times AO'} = \frac{BO}{OO' \times BO'}$ since $\frac{AO}{AO'} = \frac{BO}{BO'}$ (?), and $\frac{1}{OO'} - \frac{1}{AO'} = \frac{1}{BO'} - \frac{1}{OO'}$.

---

972. Prob.—*Given the harmonic mean and the difference between the extremes, to find the extremes.*

Sug's.—We have OO' and AB, (*Fig.* Art.) given. Then $CO \times CO' = \overline{CT}^2 = \frac{1}{4}\overline{AB}^2$, and $CO' - CO = OO'$, whence $\overline{CO'}^2 - OO' \times CO' = \frac{1}{4}\overline{AB}^2$. From this equation CO' can be constructed ( ), and the problem solved.

---

973. Theo.— *When two circles cut each other orthogonally* (*i. e.,* so that the tangents at the common point are at right angles), *any line passing through the centre of one, and cutting the other, is divided harmonically by the circumferences.*

Dem.—The tangents being perpendicular to each other pass through the centres, hence $CO \times CO' = \overline{CT}^2$. But $CB = CT$. Therefore AO is cut harmonically.

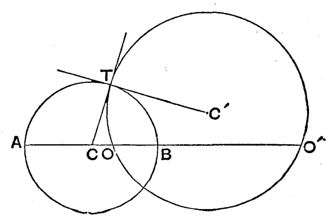

Fig. 494.

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974. Prob.—*To find the altitude of a triangle in terms of the radii of the escribed circles touching the adjacent sides.*

Solution.—Let r and r' be the radii of the escribed circles, and p the altitude. Now RT, AT, and QT are in harmonic proportion; since, considering the triangle ACT, CQ bisects its interior and RC its exterior angle (?), we have $QT : QA :: RT : RA$. But r, p, r', sustain the same relation to each other as RT, AT, QT; hence r, p, r' are in harmonical proportion. Therefore, by ( ) $p(r+r') = 2rr'$; or $p = \frac{2rr'}{r+r'}$.

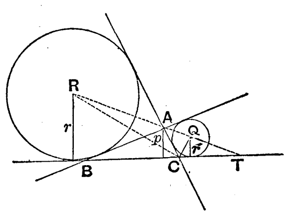

Fig. 495.

HARMONIC PENCILS.

975. Def.—***A Pencil*** of lines is a series of lines diverging from a common point.

Def.—***A Harmonic Pencil*** is a pencil of four lines cutting another line harmonically.

Ill.—In the following figure OA, OB, OC, OD constitute a *Harmonic Pencil,* since they divide the line mn harmonically at A, B, C, D.

976. Theo.—*A harmonic pencil divides harmonically every line which cuts it.*

Dem.—OA, OB, OC, OD being a harmonic pencil, that is, AB, BD, AD, being in harmonic proportion, $A'D'$, any other line cutting the pencil, is divided harmonically, so that $A'B'$, $B'D'$, $A'D'$, are in harmonic proportion. Through C and C' draw parallels to OA, as LK and $L'K'$. Now, from similar triangles, $AB : BC :: AO : CK$, and $AO : CL :: AD : CD$. But $AD : CD :: AB : BC$, since AD is harmonically divided. Hence $AO : CK :: AO : CL$, and $CK = CL$. Hence from similar triangles $C'K' = C'L'$. Again $A'B' : B'C' :: A'O' : C'K'$(?), and $A'D' : C'D' :: A'O : C'L'$ ($= C'K'$) (?), whence $A'B' : B'C' :: A'D' : C'D'$, or $A'B'$, $B'D'$, $C'D'$ are in harmonic proportion.

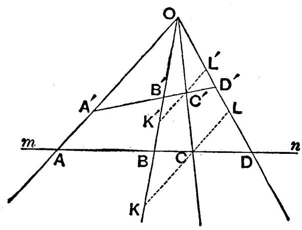

Fig. 496.

Sch.—If the line through C' cut the prolongation of AO beyond O, it is still harmonically divided; and, in fact, it is scarcely necessary to make this statement, since in all general discussions lines are to be considered indefinite, unless limited by hypothesis.

977. Def.—The alternate legs of a harmonic pencil are called *conjugate*, as OA and OC, OB and OD.

---

978. Theo.—*If two conjugate legs of a harmonic pencil be at right angles, one of them bisects the angle included by the other pair, and the other the supplement of this angle.*

Sug.—This is the converse of (   ), remembering that the bisectors of two adjacent supplemental angles are at right angles.

SECTION VI.

ANHARMONIC RATIO.

979. Def.—***The Anharmonic Ratio*** of four points in a right line is the ratio of the rectangle of the distance between the first and fourth into the distance between the second and third to the rectangle of the distance between the first and second into the distance between the second and fourth.

Fig. 497.

Ill.—The anharmonic ratio of the four points A, B, C, D is $AD \times BC : AB \times CD$.

980. The relation $AD \times BC : AB \times CD$ is expressed for brevity [ABCD].

Ill.—Thus [ABCD] means $AD \times BC : AB \times CD$; [ADBC] means $AC \times DB : AD \times BC$; [BACD] means $BD \times AC : BA \times CD$; etc. The ratio [ABCD], or $AD \times BC : AB \times CD$ is evidently the same as $\frac{AB}{BC} : \frac{AD}{CD}$.

981. Sch.—The appropriateness of the term *anharmonic* (*not*-harmonic) will be seen when we observe that, if AD is *harmonically* divided, $\frac{AB}{BC}$ *equals* $\frac{AD}{CD}$. If, therefore, $\frac{AB}{BC}$ is *not* equal to $\frac{AD}{CD}$, which is the *general* case of division, irrespective of the position of the points B and C, we may consider the *ratio* of $\frac{AB}{BC}$ to $\frac{AD}{CD}$. This general ratio, or, what is the same thing, $AD \times BC : AB \times CD$, is called the anharmonic ratio.

---

982. Theo.—*The anharmonic ratio of four points is not changed by interchanging two of the letters, provided the other two be interchanged at the same time.*

Dem. [ABCD] = [DCBA] = [BADC] = [CDAB], *i. e.*, $AD \times BC : AB \times CD = DA \times CB : DC \times BA = BC \times AD : BA \times DC = CB \times DA : CD \times AB$, which are evidently identical. [The student should notice the different segments of the line indicated by the different forms.]

983. Sch.—But [ACBD] is a *different* anharmonic ratio from [ABCD]; since $AD \times CB : AC \times BD$ is not necessarily equal to $AD \times CB : AB \times CD$. Now, as there can be twenty-four permutations of four letters, there may be formed six different anharmonic ratios from four given points in a line.

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984. Theo.—*If a pencil of four lines is cut by any transversal, the anharmonic ratio of the four points of intersection is constant.*

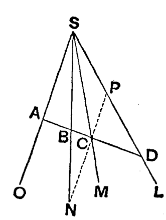

Fig. 498.

Dem. SL, SM, SN, SO, or, as we may read it, S-L,M,N,O, being such a pencil, and AD any transversal, draw through C NP parallel to SO. Then,

$$AD \times BC : AB \times CD :: \left\{ \begin{matrix} BC : AB \\ AD : CD \end{matrix} \right\} :: \left\{ \begin{matrix} CN : AS \\ AS : CP \end{matrix} \right\} :: CN : CP.$$

But $CN : CP$ is constant for all positions of C on SM. Therefore $AD \times BC : AB \times CD$ is constant for any transversal.

985. Sch.—Other constant ratios may be written from the preceding proposition and scholium. The anharmonic ratio [ABCD] is called the anharmonic ratio of the pencil. The angles of the pencil are the six angles included by the rays.

986.—Cor. 1.—*If two pencils are mutually equiangular their anharmonic ratios are equal.*

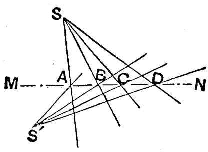

Fig. 499.

Query.—Is the converse of this corollary true?

987. Cor. 2.—*If two pencils have their intersections in the same right line, their anharmonic ratios are equal.*

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988. Def.—The anharmonic ratio of four points on the circumference of a circle is the anharmonic ratio of the pencil formed by joining these points with any point in the circumference.

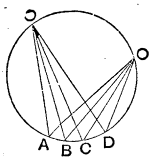

Fig. 500.

Ill.—Thus, the anharmonic ratio of the points A, B, C, D is the anharmonic ratio of the pencil O-A,B,C,D, it being immaterial where in the circumference the point O is taken, since by Cor. 1, preceding, the ratio is the same for any position of O (?).

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989. ***Theo.***—*If four fixed tangents to a circle are cut by a fifth, the anharmonic ratio of the four points of intersection, called the anharmonic ratio of the tangents, is constant.*

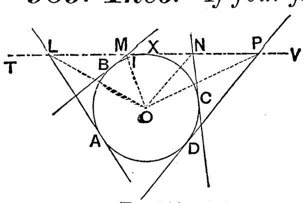

Fig. 501.

Dem. A, B, C, D being the fixed points of tangency, any transversal, as TV, cutting the tangents, has the anharmonic ratio [LMNP] constant. For the pencil O-L,M,N,P has its angles constant. Thus LOM is measured by $\frac{1}{2}$ arc (AX — IX) = $\frac{1}{2}$AB, which is constant. And in like manner MON is measured by $\frac{1}{2}$ arc BC, and NOP is measured by $\frac{1}{2}$ arc CD. Hence, by the first of the preceding corollaries, the anharmonic ratio [LMNP] is constant.

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990. The theory of anharmonic ratio is applied with great facility to the demonstration of theorems showing that several points are in a right line, and that several lines intersect in a common point. We give three specimens of each class.

991. ***Theo.***—*If two pencils have the same anharmonic ratio and a homologous ray common, the intersection of the other homologous rays are in the same right line.*

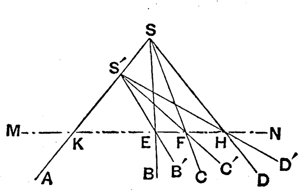

Fig. 502.

Dem.—Let S-A,B,C,D and S′-A′,B′,C′,D′ be two pencils having the same anharmonic

ratio, and the rays be SA, S′A coincident; then the intersections E, F, H are in the same right line. Let the line passing through E and F intersect SA in K, and suppose it intersect SD in H′, and S′D′ in H″. Then, since the anharmonic ratios of the two pencils are equal [KEFH′] = [KEFH″]; whence H′ and H″ are the same point, and must be the intersection of the two lines SD, S′D′, that is, H.

992. Theo.—*If in two right lines four points in the one have the same anharmonic ratio as four points in the other, and one homologous point in common, the three lines passing through the other pairs of homologous points meet in a common point.*

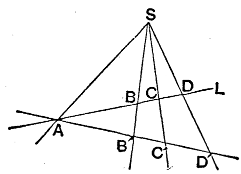

Fig. 503.

Dem.—Let A be common, and [ABCD] = [A′B′C′D′]. Draw SA and SD′. Call the point in which SD′ cuts AL D″ (for the time being). Then [AB′C′D′] = [ABCD″]. But by hypothesis [AB′C′D′] = [ABCD]. Therefore [ABCD] = [ABCD″], and D and D″ are one and the same point. Hence the three lines which pass through B and B′, C and C′, D and D′ meet in a common point S.

993. Theo.—*If the lines passing through the corresponding vertices of two triangles meet in a common point, the intersections of their homologous sides lie in the same right line.*

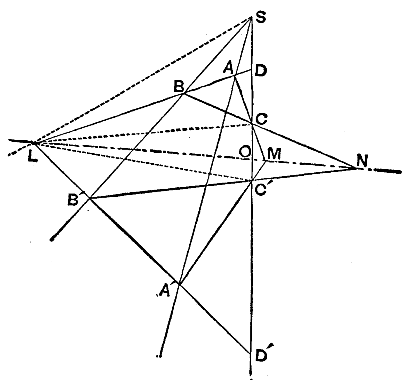

Fig. 504.

Dem.—Let ABC and A′B′C′ be two triangles so situated that the lines AA′, BB′, CC′ meet in the common point S; then L, M, N, the intersections of the homologous sides, are in a right line. For the pencil S-L, B, A, C being cut by the two transversals LD, LD′, gives [LBAD] = [LB′A′D′] (*984*). But C-L,B,A,D, and C′-L,B′,A′,D′, have these harmonic ratios, hence C-L, O,M,N, and C′-L,O,M,N, their equivalents, and having a common ray CC′, have equal anharmonic ratios, and consequently L, M, N are in the same right line, by the last proposition.

994. Theo.—*If the intersection of the corresponding sides of two triangles are in the same right line, the lines passing through their corresponding angles meet in a common point.*

Dem.—In the last figure, if AB and A′B′, AC and A′C′, BC and B′C′ have their

intersections in the same right line, as LN, the lines passing through B and B′, A and A′, C and C′ meet in a common point, as S. By (*987*) C-L,O,M,N has the same anharmonic ratio as C′-L,O,M,N, whence [LBAD] = [LB′A′D′], and the truth of the theorem follows from (992).

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995. Theo.*—The opposite sides of an inscribed hexagon have their intersections in the same straight line.*

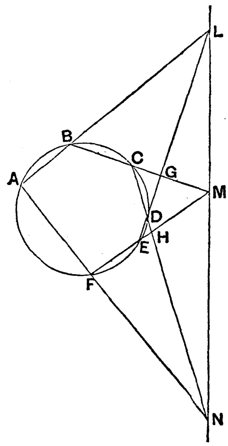

Fig. 505.

Dem.—The anharmonic ratios of the pencils B-A,E,D,C,* and F-A,E,D,C being equal ( ), LCDE, which intersects the first, is divided in the same anharmonic ratio as NHDC, which cuts the second, or [LCDE] = [NCDH]. But these lines have a common homologous point D, hence the lines joining the other pairs of homologous points, as LN, CG, EH, meet in a common point, as M. Therefore L, M, N are in the same right line.

996. Sch.—This theorem is due to Pascal, whose wonderful achievements in his brief life of thirty-nine years (1623–1662) have been the admiration of all succeeding generations.

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997. Theo.*—The diagonals joining the opposite vertices of a circumscribed hexagon intersect in a common point.*

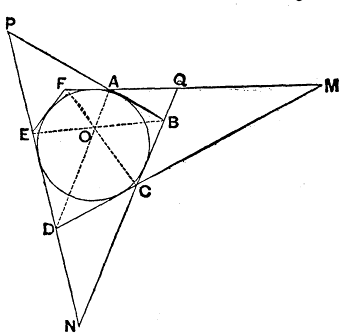

Fig. 506.

Dem.—Consider AB, BC, CD, EF four fixed tangents cut by ED and FA. Then [PNDE] = [AQMF] (*989*). Hence the anharmonic ratios of B-P,N,D,E,* and C-A,Q,M,F are equal ( ); and since they have a common ray (CQ, BN) the intersections A, O, D, of their homologous rays, are in the same right line. Therefore the diagonals pass through a common point.

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* The student can *conceive* the rays BE, BD, etc,, as drawn, without encumbering the figure with them.

SECTION VII.

POLE AND POLAR IN RESPECT TO A CIRCLE.

998. Def.—If a secant to a circle be revolved about a fixed point in the plane of the circle, the locus of the harmonic conjugate of the fixed point, in reference to the intersections, is the ***Polar*** of the fixed point. The fixed point is the ***Pole*** of the *Polar Line.* The terms pole and polar as here used are correlative, and neither has any significance without the other.

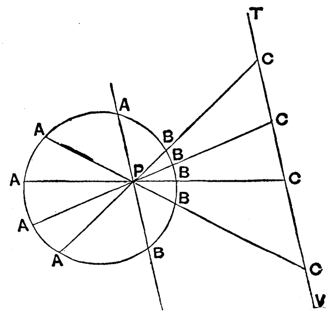

Fig. 507.

Ill.—Let AP be a secant revolving about the fixed point P, and let C be so taken that (in every position) AC : CB : : AP · BP, then is the locus of C the *Polar* of P, and P is the *Pole* of the locus of C.

---

999. ***Theo.***—*The Polar of a given point in respect to a circle is a right line.*

Dem.—Let P be the pole, AP any secant passing through P, and a point in the polar. The locus of C is required. Draw PL through the centre, and let fall the perpendicular CC′. Draw AL, AH, AC′F, and C′B. Since AC : CB : : AP : BP, C′P bisects the angle BC′F, the exterior angle of the triangle AC′B (?); hence, as LAH is a right angle, AL bisects NAC′, the exterior angle of the triangle C′AP (?). Therefore, PL is harmonically divided at C′, and H; and, C′ being a fixed point, and C any point in the locus, the locus is the perpendicular TCC′V.

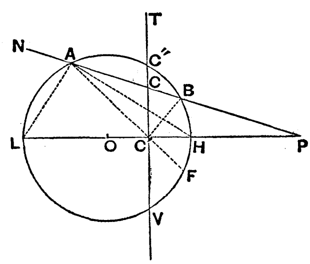

Fig. 508.

1000. Cor. 1.—*Since, as the secant revolves, the points* A *and* B *will vanish in* C″, C″ *is the point at which a tangent from the pole* P *touches the circle.*

1001. Cor. 2.—*Drawing* OC″, C″P, *we see that* $\overline{OC''}^2$ (*or* $\overline{OH}^2$ $= OC' \times OP$.

1002. Cor. 3.—*The polar of a point in the circumference is a tangent at that point.*

For, as $OC' \times OP$ is constant and equal to $\overline{OH}^2$, OP diminishes as OC' increases, and when $OP = OH$, $OC' = OH$ also.

1003. Prob.—*To draw the polar to a given pole in respect to a given circle.*

Cor. 1 effects the solution.

1004. Prob.—*To find the pole of a polar to a given circle.*

Through the centre draw a perpendicular to the polar. [The student should be able to complete the solution.]

1005. Def.—The point C' where the polar cuts the line passing through the pole and the centre of the circle is called the *Polar Point.*

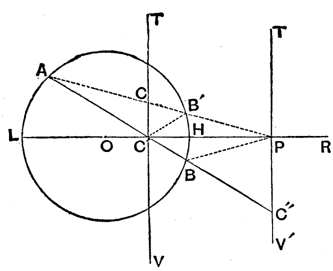

Fig. 509.

1006. Theo.—*The pole and polar point are interchangeable.*

Dem.—TV being the polar to P, we are to show that $T'V'$, parallel to TV and passing through P, is polar to C'; *i.e.*, that any secant, as $AC'C''$, passing through C', is divided harmonically in the intersections with the circumference, C', and the intersection with $T'V'$. Drawing AP, since P is the pole of TV, we have, as in the last demonstration, angle APB bisected by PC'; and consequently RPB bisected by PC''. Therefore $AC' : C'B :: AC'' : BC''$. Q. E. D.

1007. Theo.—*The polars of all the points in a right line pass through the pole of that line; and, conversely, The poles of all straight lines which pass through a given point are in the polar of that point.*

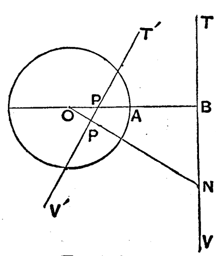

Fig. 510.

Dem.—1st. TV being a given line and P its pole, we are to show that the polar to any point, as N, passes through P. Draw through P a perpendicular to ON; then P' is the polar point to N. For, $OP : ON :: OP' : OB$ (?): whence $ON \times OP' = OP \times OB = \overline{OA}^2$. Therefore, $T'V'$ is the polar of N (?). 2nd. P being any point and TV

its polar, the pole of any line, as T′V′ passing through P, is in TV, as at N. Draw ON perpendicular to T′V′. Then, as before, $ON \times OP' = OP \times OB = \overline{OA}^2$, and N is the pole of T′V′.

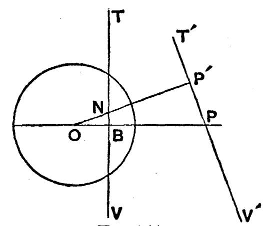

Fig. 511.

1008. Cor.—*The pole of a straight line is the intersection of the polars of any two of its points ;* and, conversely, *The polar of any point is the straight line joining the poles of any two straight lines passing through that point.*

RECIPROCAL POLARS.

1009. Def.—If two polygons be constructed, one within, or inscribed in, a circle, and the other without, or circumscribed about the same circle, such that the vertices of the one are the poles of the sides of the other, the two polygons are called ***Reciprocal Polars*** ; and the circle is called the *Auxiliary Circle.*

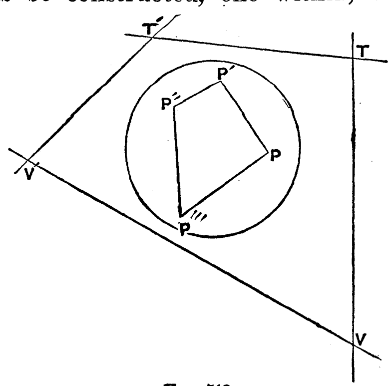

Fig. 512.

The possibility of constructing such polygons is apparent from the last theorem. When the points P, P′, P″, P‴ are in the circumference, TV, TT′, T′V′, V′V become tangents, as appears from (1002).

1010. ***Prob.***—*Having given one of two reciprocal polars, to construct the other.*

The student should be able to make the construction.

1011. By means of the relation between reciprocal polars a large class of propositions relating to the relative positions of lines and points, become, as it were, double ; *i.e.*, one proposition being proved, another can be inferred. The process by which the inference is made is called *reciprocation.* We will give an example.

1012. Prob.—*To deduce the reciprocal of Pascal's theorem* (995).

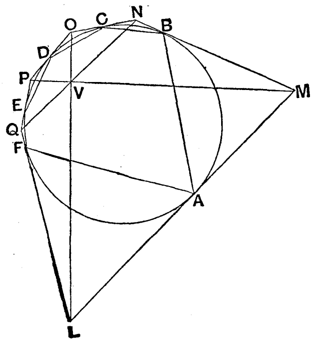

FIG. 513.

SOLUTION.—Draw tangents at the six vertices of the inscribed hexagon. Thus, a circumscribed hexagon is formed whose sides are the poles of the vertices of the inscribed hexagon, through the vertices of which they respectively pass (1002). Now, drawing the diagonals PM, NQ, OL, they are the polars of the intersections of the opposite sides of the inscribed hexagon, as PM, polar to the intersection of DE and CB (?); and hence they pass through a common point, as V. Thus we have Brianchon's theorem, viz.: *The lines joining the opposite angles of a circumscribed hexagon pass through a common point.*

1013.—The three following theorems are of frequent use in applying the theory of reciprocation.

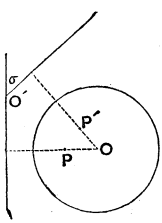

FIG. 514.

1014. Theo. — *The angle included by two straight lines is equal to the angle included by the lines joining their poles to the centre of the auxiliary circle.*

DEM.—The pole of a line being in the perpendicular from the centre of the auxiliary circle upon the line (1000), O′ is the supplement of O; hence $o = O$.

1015. Theo.—*The distances of any two points from the centre of the auxiliary circle are to each other as the distances of each point from the polar of the other.*

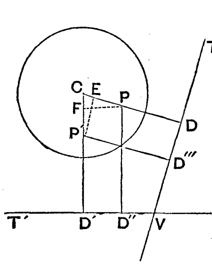

FIG. 515.

DEM. P, P′ being the points, and TV, T′V their polars respectively, we are to show that

$$CP : CP' :: PD'' : P'D'''.$$

By (   ) $R^2 = CP \times CD = CP' \times CD'$, R being the radius of the auxiliary circle. Whence $CP : CP' :: CD' : CD$. But $CP : CP' :: CF : CE$ (?), and there follows $CF : CE :: CD' : CD$, $CD' - CP$ $(= P'D''')$ $: CD - CP$ $(= PD'') :: CF : CE :: CP : CP'$.

SCH.—This is known as *Salmon's Theorem.*

1016. Theo.—*The anharmonic ratio of four points in a straight line is equal to that of the pencil formed by the four polars of these points.*

Dem.—1st. The polars pass through a common point and thus form a pencil (?). 2d. The angles included by the lines joining the four points with the centre, and those included by the polars are equal (?), hence the two pencils have the same anharmonic ratio.

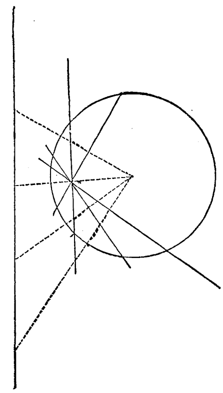

Fig. 516.

SECTION VIII.

RADICAL AXES AND CENTRES OF SIMILITUDE OF CIRCLES.

1017. Def.—***The Power of a Point*** in the plane of a circle is the rectangle of the distances from the point to the intersections of the circumference by a line passing through the point.

Ill.—Thus, the *power of a point* P, without the circle, is PA × PB;—the power of a point within, as P′, is P′A′ × P′B′; the power of a point in the circumference is *zero*, since one of the distances is then O;—the power of the centre is the square of the radius.

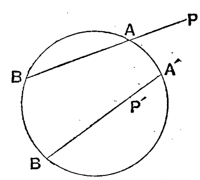

Fig. 517.

1018. Cor.—*The power of a given point with respect to a given circle is a constant quantity.*

Thus PA × PB = the square of the tangent from P to the circle, in whatever position PB lies, so long as it passes through P. So also P′A′ × P′B′ = the rectangle of the segments of any other chord passing through P′.

1019. Def.—***The Radical Axis*** of *Two Circles* is the locus of the point whose powers with respect to the two circles are equal.

1020. Prop.—*The Radical Axis of two circles is a right line.*

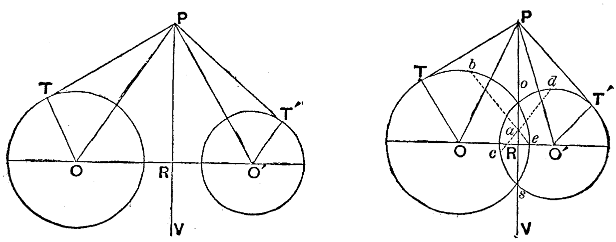

Fig. 518.

Dem.—Join the centres of the two circles O, O′, and take a point R on this line such that $\overline{OR}^2 - \overline{O'R}^2 = \overline{OT}^2 - \overline{O'T'}^2$, or $\overline{OR}^2 - \overline{OT}^2 = \overline{O'R}^2 - \overline{O'T'}^2$, and erect PR perpendicular to OO′. Then P being *any* point in this perpendicular, $\overline{OP}^2 - \overline{OR}^2 = \overline{O'P}^2 - \overline{O'R}^2$. Adding this to the preceding equation, we have $\overline{OP}^2 - \overline{OT}^2 = \overline{O'P}^2 - \overline{O'T'}^2$, or $\overline{PT}^2 = \overline{PT'}^2$. $\therefore$ PT = P′T′, PT and PT′ being tangents to the circles from any point in PR. Hence PV is the radical axis of the two circles.

1021. Cor.—*When the circles do not intersect, the Radical Axis lies between them, touching neither; when they are tangent, either externally or internally, the radical axis is the common tangent; when they cut each other, the axis of the common cord produced.*

1022. Sch.—When the circles intersect it might seem that the above demonstration fails for points within, as in the common chord. But, the powers of any point in this chord are still equal. Thus, at the intersections the powers are zero; and at any other point in the chord, as a, $ab \times ac = ad \times ae$, since each is equal to $ao \times os$.

1023. Cor.—*There is an infinite number of circles having their centres in the same right line, which have the same radical axis as any two given circles.*

Thus, in the first figure, PV being the radical axis of the circles O, O′, letting circle O remain fixed, O′ may vary indefinitely so that $\overline{O'R}^2 - \overline{O'T'}^2$ remains constant, and equal to $\overline{OR}^2 - \overline{OT}^2$.

1024. Prob.—*Given two circles, to draw their radical axis.*

Solution.—In any case, draw a common tangent, bisect it, and through the point of bisection draw a line perpendicular to the line joining the centres. When the circles are tangent to each other, the distance between the points of tangency is O; hence the perpendicular is erected at this point. When they intersect, produce the common chord, or use the first method.

1025. Prop.—*When two circles cut each other orthogonally, that is, at right angles, the square of the radius of either is equal to the power of its centre with respect to the other.*

DEM.—The power of O with respect to circle O′ is $Oa \times On = \overline{OP}^2$, and of O′ with respect to circle O, $O'b \times O'm = \overline{O'P}^2$; since, as the circles cut each other orthogonally, their tangents are at right angles, and the tangent to either passes through the centre of the other.

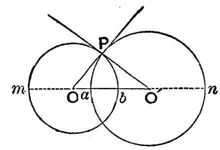

FIG. 519.

1026. Prop.—*The radical axes of a system of three circles whose centres are not in the same straight line, intersect at a common point.*

DEM.—Since O, O′, O″ are not in a straight line, the radical axes of O, O′, and O, O″, as PV″ and PV intersect. Let P be their common point. Now the power of P with respect to O′ is equal to its power with respect to O″, since each is equal to its power with respect to O. Hence P is a point in the radical axis of O′, O″.

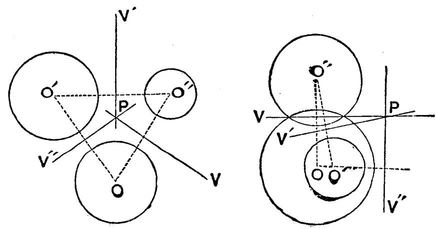

FIG. 520.

1027. COR.—*If the centres are in the same straight line, their radical axes are parallel, and the common point is at infinity.*

1028. DEF.—The intersection of the radical axes of three circles is called their ***Radical Centre.***

CENTRES OF SIMILITUDE.

1029. DEF.—If the line joining the centres of two circles be divided externally, as at C, and internally, as at C′, in the ratio of the radii, these points are respectively the *External* and the *Internal* ***Centres of Similitude*** of the two circles.

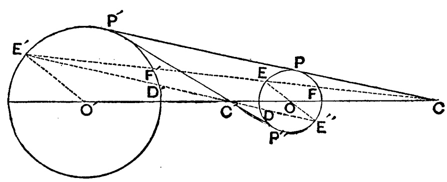

FIG. 521.

ILL.—If $CO : CO' :: EO : E'O'$, C is the external centre of similitude; and, if $C'O : C'O' :: EO : E'O'$, C′ is the internal centre of similitude.